UNIT 3
SPECIALIST MATHS

SUMMER

SCHOOL

VCE SUMMER SCHOOL

Unit 3 Specialist Mathematics

Area of Study 1

Discrete Mathematics

Area of Study 2

Functions, Relations & Graphs

Area of Study 3

Algebra, Number & Structure

Area of Study 4

Calculus

VCE Accreditation Period
2023 – 2027

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VCE SUMMER SCHOOL HEAD START LECTURES

STUDY DESIGN (2023 – 2027) – EDITION 1

ACKNOWLEDGEMENTS

TSFX wishes to thank the Victorian Curriculum and Assessment Authority (VCAA) for permission to reproduce extracts from their publications.

ABOUT THE COVER IMAGE

THE POWER OF ART

Engaging with art is essential to the human experience. Almost as soon as motor skills are developed, children communicate through artistic expression. Throughout each stage of our lives, art plays different and important roles. The arts have the power to bring joy, stir up emotions and influence our behaviour. Art crosses all divides. It breaks down cultural, social and economic barriers and plays a big role in how humans see and interact with others, and the world in general.

Art decreases stress levels and improves mental health and well-being, particularly in patients suffering chronic or terminal illness. It has the power to educate people and convey meaning in a way that can be appreciated by every person. Furthermore, it gives us the opportunity to travel through time and learn from the beliefs, dreams, habits, thoughts, culture and lives of people in different places and times.

The arts also challenge us with different points of view, encourages communication, promotes stronger critical thinking and problem-solving skills and unlocks the potential of the human mind. It is also closely linked to academic achievement, civic engagement and social and emotional development.

The benefits of art are significant and undeniable. Use it to benefit both your mental and physical health as you journey through your VCE.

If you believe there is an error in this publication, please contact us as soon as possible (<u>admin@tsfx.edu.au</u> or 966 333 11). Amendments, errata, revisions, clarifications and further discussions will be published on our website at <u>www.tsfx.edu.au/errata</u>.

SECTION 1: UNIT 3 SPECIALIST MATHEMATICS COURSE OUTLINE & ASSESSMENT (2023 – 2027)

UNITS 3 AND 4: SPECIALIST MATHEMATICS

ACCREDITATION PERIOD: 2023 – 2027

INTRODUCTION

Specialist Mathematics is studied in conjunction with Mathematical Methods and it expands and relies on the content covered in Mathematical Methods.

TOPICS

- Logic and Proof
- Complex Numbers
- Vectors
- Trigonometric Functions
- Functions and Graphs
- Rational Functions
- Differential Calculus
- Integral Calculus
- Differential Equations
- Kinematics
- Vector Calculus
- Probability and Statistics

REQUIREMENTS

The main form of assessment throughout the year is via SACs (School Assessed Coursework), internally set and assessed, externally moderated. SACs contribute 40% to the Study Score.

This means that there will be some form of assessment occurring throughout various periods of the year.

For each unit you will be required to demonstrate the achievement of three outcomes.
Your school will decide how they will judge achievement. (This is to get an S.)

The assessments for Specialist Mathematics will constitute:

UNIT 3

One Application Task worth 50 marks.

This task should take 4 - 6 hours over a period of 1 - 2 weeks, completed mainly during class time. This assessment will constitute an investigative task with topics from at least two areas of study and with three components of increasing complexity.

UNIT 4

Two Modelling or Problem Solving Tasks, each worth 25 marks.

Each task should take 2 – 3 hours over a period of 1 week and are completed mainly during class time.

One of the Analysis tasks must be related to the Probability and Statistics areas of study.

EXAMINATIONS

Examination 1 (November) Technology Free Exam (20% of Study Score)

Length: 1 hour exam with 15 minutes reading time (40 marks).

Content: Short answer and some extended answer questions worth 40 marks.

Formula sheet provided.

No calculators or notes

Examination 2 (November) Technology Active Exam (40% of Study Score)

Length: 2 hour exam with 15 minutes reading time.

Content: Multiple choice questions and Extended answer questions.

(80 marks: 20 multiple choice questions and 60 marks extended answer questions).

Formula sheet provided.

CAS calculator with full memory capacity, plus a scientific calculator if desired.

One bound reference, text (annotated) or lecture pad may be brought into the exam.

IMPORTANT NOTE

Throughout this book, you will observe the following symbol: 

This symbol means “by hand" and is used to represent technology free questions i.e. do not use any form of technology when working through these questions. However, you can use technology to check your answers; it can help you to use your technology more efficiently, and gain familiarity with it, when you are allowed to use one.

You will also observe another symbol: 

This symbol indicates that technology is allowed, so make the most out of your CAS calculator. There is no substitute for the year-long practice with your CAS calculator – don’t leave this until the end of the year.

THE MATHEMATICS REFERENCE MATERIALS

CRITERIA FOR REFERENCE MATERIALS

Acceptable "Reference Materials" include textbooks or your own prepared permanently bound notes.

For example:

- A securely bound lecture pad.
- A single spine, bound student-constructed set of notes without foldouts or sticky tabs. These notes may contain any combination of handwriting, typed, scanned or photocopied notes
- An exercise book with cloth, glue or staple binding.

ONLY ONE bound set of notes is allowed into the above-mentioned examinations.

(a) Materials must be in book format of A4 size or smaller when closed.

(b) All materials must be of uniform size throughout the book.

(c) There is no page limit for the Reference Materials. As long as the materials are contained in one book or bound item – they will be accepted.

(d) Materials must be held together via a single horizontal or vertical spine.

(e) Pages must be permanently bound and securely attached to the spine. They must not be able to be detached from the Reference Materials during the exam.

Basically, you cannot use Reference Materials that are presented in a form that will enable you to modify the content by adding or removing pages or parts of pages.

Forms of collation / binding that are not acceptable include:

- Ring-binder folders.
- Plastic A4 slips (permanent or removable) into which pages may be inserted or removed.
- Manila and similar folders with clip, clamp, slide and metal prong style binding of loose-leaf material.
- Glued lecture pads.
- Bound books that have perforations designed so as to allow pages to be detached.
- Coloured tabs that highlight sections that could drop off in an exam. (if you want to use this idea then a whole coloured page with the tabs as part of the cardboard needs to be inserted as part of the spine).

If one or more pages (or part of a page) can be or are detached from the Reference Materials, the entire reference materials will be confiscated by the examination supervisor, and the incident will be reported to the VCAA as a breach of rules.

Acceptable forms of Reference Materials:

A4 (or smaller) sized spiral bound exercise books, lecture pads that have been permanently bound by stapling or some other means, an exercise book with cloth, glue or staple binding or A4 sheets of paper that have been appropriately bound.

Materials can be bound using the typical equipment located in an "Officeworks" or similar store.

- No fold outs, maps or brochure style components are permitted.
- You cannot attach removable tabs, post-it notes or other items to the reference materials.
- You are allowed to firmly attach (e.g. by glue or adhesive tape) additional materials to the pages in the bound reference, as long as all four corners and side edges of these additional notes are securely pasted.
- Reference Materials can be annotated (have your handwritten comments/additional notes written in).
- It is helpful to have some spare pages left at the end of the book so that, after binding, some late notes can be added.
- You are allowed to draw attention to sections of the Reference Materials by folding or cutting page corners, or by colour coding pages.
- Students are allowed to insert coloured dividers into their prepared materials as long as these are also part of the 'spine'.
- The rules relating to acceptable "Reference Materials" continue to be refined and clarified by the VCAA. We therefore recommend that students regularly refer to the VCE Bulletin (produced by the VCAA) for changes and clarifications. These bulletins can be accessed at **www.vcaa.vic.edu.au**

OUR ADVICE TO STUDENTS

The "open book" style of testing in the mathematics subjects has resulted in more difficult questions and applications in Examination 2.

These examinations are likely to emphasise applications that are **NOT** typically covered in textbooks, meaning that students will need to possess strong mathematical skills/techniques, as well as significant experience in advanced applications if they are aiming for the higher scores.

As Examination 2 is more than likely to emphasise questions and applications that are **NOT** typically covered in textbooks, we recommend that students prepare their own Reference Materials. These are typically written throughout the year in class as notes are given. Some students like to rewrite their notes before the exams, but in our opinion the notes, in the order they were written, is a useful memory guide as you have prepared them yourself throughout the year.

What Sort of Questions are Likely to Appear in the Examinations?

Examination 1 is quite straight forward but requires that students possess strong algebraic and arithmetic skills (including fractions and surds), as well as excellent graphing skills.

The multiple-choice questions in Examination 2 are usually written so that the use of a CAS calculator is not necessarily a benefit, although the calculator is used sometimes in straightforward questions. The calculator also can be used sometimes to ‘guess and check’, but this is usually a longer method and a last resort.

A condition for answers to short answer and extended answer questions is that they be given in exact values unless otherwise specified. This means that the default answer is the exact one. The other condition is that if more than 1 mark is allocated to a question then appropriate working must be shown, which means that students are required to show working steps in order to obtain full marks. Furthermore, the extended answer questions incorporate applications that are not typically detailed in textbooks, and will require students to apply their knowledge of taught mathematical principles to applications that students may not have been exposed to.

As an example, students may be given, in a multiple-choice question, an integral to evaluate. A CAS calculator can simply evaluate this integral in one step, so the question will be given in the form of the ***u*** substitution (change of variable) that would be required to interpret and evaluate the integral by hand.

PREPARING REFERENCE MATERIALS

The fact that the Reference Materials have no page limit and no restrictions in content implies that the VCAA is confident that no amount of notes or resources will be of significant benefit or advantage to those students who do not know their course materials.

So, what should students include in the mathematics Reference Materials in order to make these notes powerful resources in the end of year exam?

Your materials should include the following:

- Summaries of vital concepts and rules.
- Guidelines as to when each rule can be applied.
- Step by step methodologies with sample calculations that you can fall back on if you develop “mind blanks” in the examination.
- Limitations of rules and methodologies.
- Exceptions to rules and methodologies.
- Flow charts that describe how to tackle complex analysis style questions.
- Summaries of every technique that was covered in the examinable and prerequisite units and what each technique is used for. This summary will serve as a valuable prompt that may assist students to determine which methods or techniques are required to solve those questions they have not been previously exposed to.

- Details of all the common errors made by students, prompts, exam tricks and watch-outs under each topic of the course. Read through each appropriate “Tricks and Watch-out List” before attempting questions on the examination. This will ensure that valuable marks are not lost during the course of the examination.

  For example:

  The Logarithmic Functions section of your materials should include a reminder to “check the validity of answers” before leaving logarithmic questions. Many students lose marks in this topic as they forget to substitute the calculated values of x into the given equation so as to determine whether the answers are defined (the log of zero or a negative number is undefined). In fact – you should check the validity of all answers as anomalies may arise with every function type.

  For example:

  If the question asks for **coordinates**, don’t just give the x value. This is a very common way of losing marks.

- Sample analysis examination questions that have been used in recent year’s CAS-active examinations.

- CAS calculator methodologies and your usual syntax for the brand of CAS you are using.

DETAILED INSTRUCTIONS

As you are studying each topic in mathematics this year, write, type, or scan summary notes, methodologies (in a step-by-step fashion) as well as worked examples into a WORD document. This will allow you to edit and update your notes without having to re-write large slabs of materials before the end of the year examinations. (Remember to leave a large enough margin so that you do not lose part of the notes when you bind the final product.)

In your notes, include worked examples from text books (to protect you from mind blanks and anxiety attacks), challenging questions that you come across throughout the year (including past analysis exam questions), a clear and detailed index that will enable you to quickly and easily locate information, watch-outs and common errors lists (to protect you from losing valuable marks in the exam) as well as guidelines and examples relating to solving the difficult and strange applications that are not detailed in your text books (this will require a bit of research).

We suggest that students structure their Reference Materials in two different sections.

The first section should contain the theory, watch-outs, methodologies, prompts and reminders for each topic. Worked examples supporting or illustrating the concepts/watch-outs in Section 1 should all be placed into a separate section or a Question Bank.

The questions in the Question Bank do not need to be organised in any particular fashion. As you come across a new question – just add it to the end of the Bank and annotate your Theory Section with the relevant Question number, so that you may quickly and easily locate worked examples when necessary.

For example: If required to find a derivative of a hybrid or absolute value function:

Step 1: Write the rules to describe the various parts of the domain.

Step 2: Find the derivative of each individual rule making sure to adjust the domain of the derivative function.

Students may also highlight each section of their notes using different coloured text.

For example:

Theory could be typeset using red text, general methodologies in blue text and common errors and watch-outs in green text, so that appropriate materials can be located quickly and with ease.

Regularly review and update your materials to incorporate changes and additional worked examples that you have encountered throughout the year.

Very importantly – Refer to your Reference Materials regularly in the weeks leading up to the examinations so that you can learn to locate relevant sections quickly (you will not have time to search for materials during the exam).

BEFORE THE EXAMINATIONS

You may choose to print each topic on different coloured paper so that relevant materials can be quickly located – but only use pastel coloured paper so that the notes are easy to read.

Alternatively, use coloured A4 dividers to separate topics.

Some students find it useful to cut into the first page of each new section and colour it in a different colour for each section.

We further recommend that students print two pages of notes onto one side of a sheet of paper so that materials are not bulky, and so that sections of interest can be located with speed. Insert any loose A4 sheets containing additional questions/materials into the appropriate section of your notes and then bind the notes together and carefully so that no more pages can be added or removed.

Some schools require that students have their notes checked and signed off by their teachers in the weeks prior to the actual examination.

SECTION 2: LOGIC AND PROOF

AREA OF STUDY 1: DISCRETE MATHEMATICS

LOGIC AND PROOF

(This is a new topic in the 2023-2027 Specialist Mathematics course.)

LOGICAL STATEMENTS

- An example of a logical statement is: *if it is raining then there are clouds in the sky*.
- Logical statements that we might encounter in Specialist Mathematics often have the form: if … then …

If P represents the statement "it is raining" and Q represents the statement "there are clouds in the sky" then we could write $P \Rightarrow Q$ to represent the statement "if it is raining then there are clouds in the sky".

The statement "if it is raining then there are clouds in the sky" represented by $P \Rightarrow Q$ seems reasonable and so we say that it is true. Reversing the direction of the implication sign may not always result in a true statement:

$Q \Rightarrow P$ means "if there are clouds in the sky then it is raining". What do you think about this statement?

PROOF BY CONTRAPOSITIVE

The **contrapositive** of the statement $P \Rightarrow Q$ is made by **negating** the statements P and Q and reversing the order. That is:

The contrapositive of $P \Rightarrow Q$ **is** $(not\, Q) \Rightarrow (not\, P)$

If P represents the statement "it is raining" then $(not\, P)$ represents the statement "it is **not** raining. If Q represents the statement "there are clouds in the sky" then $(not\, Q)$ represents the statement "there are **no** clouds in the sky".

So the contrapositive statement (in words) is: "If there are no clouds in the sky then it is not raining". This statement seems entirely reasonable as well.

In fact, the statements $P \Rightarrow Q$ and $(not\, Q) \Rightarrow (not\, P)$ are logically equivalent. This means that there are wither both true or both false.

The contrapositive can be very useful is proving mathematical statements.

For example:

Suppose $x \in \mathbb{Z}$. Prove that if $3x+5$ is even then x is odd.

We will show how to do this by a **direct proof** and by a **contrapositive proof**.

DIRECT PROOF:

If $3x+5$ is even then $3x+5=2n$ for some integer n. That is:

$$\begin{aligned} &3x+5=2n \\ &\Rightarrow x=2n-2x-5 \\ &\qquad = 2\left(n-x-3\right)+1 \end{aligned}$$

We have shown that if $3x+5$ is even then x is odd.

CONTRAPOSITIVE PROOF:

The contrapositive statement is "if x is even then $3x+5$ is odd". Let $x=2n$.
Then:

$$\begin{aligned} 3x+5 &= 3(2n)+5 \\ &= 2(3x+2)+1 \end{aligned}$$

We have shown that if x is even then $3x+5$ is odd.

QUESTION 1

(a) Prove that if x^2+4x-3 is even then x is odd.
Hint: Prove the contrapositive statement

(b) Prove that if x^3-2x+1 is odd then x is even.

PROOF BY CONTRADICTION

Suppose we want to prove the truth of a statement P. If by assuming that P is false leads to a contradiction then we have established that P is in fact true.

QUESTION 2

(a) Prove that $\sqrt{2}$ is irrational.

We begin by assuming that $\sqrt{2}$ is rational. That is, $\sqrt{2} = \frac{a}{b}$ where $a, b \in \mathbb{Z}$ and $\frac{a}{b}$ is fully simplified (that is, all common factors are cancelled). We begin by squaring both sides:

$$\sqrt{2} = \frac{a}{b}$$
$$\Rightarrow 2 = \frac{a^2}{b^2}$$
$$\Rightarrow a^2 = 2b^2$$

This establishes that a^2 is even and so a is even. Therefore, $a = 2m$ where $m \in \mathbb{Z}$. Therefore:

$$(2m)^2 = 2b^2$$
$$\Rightarrow 4m^2 = 2b^2$$
$$\Rightarrow 2m^2 = b^2$$

This shows that b^2 is even and so b is even. But it cannot be the case that both a and b are even. We have a contradiction. Therefore, $\sqrt{2}$ is not rational. It is irrational.

(b) Prove that there are infinitely many prime numbers.

QUESTION 3

Suppose $m, n \in \mathbb{Z}$ and $m + n > 73$. Prove that $m > 37$ or $n > 37$.

Solution

QUESTION 4
Prove that $\log_3 10$ is irrational.

Solution

COUNTEREXAMPLES

Sometimes, it is possible to find a counterexample to a statement, therefore showing the statement to be false.

For example, the statement “all rectangles are squares” is false. In order to demonstrate the falsity of the statement, all I need to do is to produce a rectangle which is not a square.

QUESTION 5
Consider the statement: “All prime numbers are odd”.
Which **one** of the following numbers could be used to disprove this statement?

A. 0
B. 1
C. 2
D. 3
E. 4

QUESTION 6
Consider the statement: “For any real numbers x, y, z , $x \le y \Rightarrow zx \le zy$”.
Which **one** of the following can be used as a counterexample to disprove this statement?

A. $x=0,\ y=0,\ z=0$
B. $x=-3,\ y=-2,\ z=4$
C. $x=-6,\ y=-1,\ z=-2$
D. $x=0,\ y=5,\ z=3$
E. $x=-1,\ y=-1,\ z=-1$

PROOF BY MATHEMATICAL INDUCTION

Consider the following statement (or theorem) about the sum of the first n natural numbers:

$$1+2+3+\ldots+n=\frac{n(n+1)}{2}$$

There are various ways to prove this result. If you have studied some sequences and series in the past, then you have the necessary tools to establish this result.

However, we will use **Mathematical Induction** to prove this statement.

METHOD:

A proof by mathematical induction (or just "induction") will work like this:

Let $P(n)$ be some proposition about the natural number n.

Step 1: Show that $P(1)$ is true. (Note that in some situations we won't start at 1.)
This is called the **base step.**

Step 2: Assume that $P(k)$ is true for some $k>1$.
(Again, this number won't necessarily be 1.)

Step 3: Show that $P(k)\ true \Rightarrow P(k+1)\ true$.
This is the **induction step.**

QUESTION 7

Prove by induction that $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ for all $n\geq 1$.

Solution

Let $P(n)$ be the proposition that $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ for all $n\geq 1$.

Step 1: $P(1):\ 1=\frac{1(1+1)}{2}=1$. So $P(1)$ is true.

Step 2: Assume that $P(k)$ is true for some $k>1$. That is: $1+2+3+\ldots+k=\frac{k(k+1)}{2}$

Step 3: Consider the statement $P(k+1)$:

$$\begin{aligned} LHS\ of\ P(k+1) &= \underbrace{1+2+3+\ldots+k}_{=\frac{k(k+1)}{2}}+k+1 \\ &= \frac{k(k+1)}{2}+k+1 \\ &= \frac{k(k+1)+2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= RHS\ of\ P(k+1) \end{aligned}$$

So $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, $1+2+3+\ldots+n=\frac{n(n+1)}{2}$ for all $n\geq 1$.

QUESTION 8

Let $E(n)$ be the sum of the first n even numbers:

$E(1) = 2$, $E(2) = 2+4$, $E(3) = 2+4+6$, …

Which **one** of the following gives a proof of the statement "$E(n) = n(n+1)$ for $n = 1, 2, 3, \ldots$" using mathematical induction?

A. Show that $E(1) = 2$, assume that $E(n) = n(n+1)$ and deduce that $E(n+1) = (n+1)(n+2)$.

B. Show that $E(1) = 2$, assume that $E(n) = 2+4+6+\ldots+2n$ and deduce that $E(n+1) = 2+4+6+\ldots+(2n+1)$.

C. Show that $E(1) = 2$, assume that $E(n) = 2+4+6+\ldots+2n$ and deduce that $E(n+2) = 2+4+6+\ldots+(2n+2)$.

D. Show that $E(1) = 2$, assume that $E(n) = n(n+1)$ and deduce that $E(n+2) = (n+1)(n+2)$.

E. Show that $E(n+1) = E(n) + 2n$

Solution

SUMMATION AND PRODUCT NOTATION

In Specialist Mathematics 1 & 2 you may have studied sequences and series. Recall that a series is the sum of the terms of a sequence. In particular:

If $a_1, a_{2,3}, \ldots, a_n$ **is a sequence, then** $a_1 + a_2 + a_3 + \ldots + a_n$ **is a series**

We can write the series in a compact form:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n$$

The symbol i is the **index**. This symbol may change from problem to problem.
Typical symbols for the index are i, j and k.

The **summation notation** $\sum_{i=1}^{n} a_i$ simply says to add up the terms of the sequence, starting at a_1 and stopping when a_n is reached.

The **product notation** is as follows: $\prod_{i=1}^{n} a_i = a_1 \times a_2 \times a_3 \times \ldots \times a_n$

For example, the familiar factorial $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ could be written in the form:

$$n! = \prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \ldots \times n$$

QUESTION 9

Prove by mathematical induction that $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$.

Solution

QUESTION 10

Prove by mathematical induction that $\sum_{k=1}^{n} k^2 > \frac{(n-1)^3}{3}$.

Solution

QUESTION 11

Prove by mathematical induction that $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1}$.

Solution

QUESTION 12

Prove by mathematical induction that $\prod_{i=2}^{n}\left(1-\frac{1}{i^2}\right)=\frac{n+1}{2n}$ for all $n \geq 2$.

Solution

DIVISIBILITY STATEMENTS

Mathematical induction can be used to prove statements about factors in expressions.

Consider the following example.

QUESTION 13

Use mathematical induction to prove that 3 is a factor of $n^3 - n + 3$ for all $n \geq 1$.

Solution

Let $P(n)$ be the proposition that 3 is a factor of $n^3 - n + 3$ for all $n \geq 1$.

$P(1)$ is true as 3 is a factor of $1^3 - 1 + 3 = 3$.

Suppose that $P(k)$ is true. That is, 3 is a factor of $k^3 - k + 3$. In other words, $k^3 - k + 3 = 3m$ for some $m \in \mathbb{N}$.

Consider $P(k+1)$**:**

$$\begin{aligned}(k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 3k + 1 - k - 1 + 3 \\ &= \underbrace{k^3 - k + 3}_{=3m} + 3k^2 + 3k \\ &= 3\left(m + k^2 + k\right)\end{aligned}$$

3 is clearly a factor of $3\left(m + k^2 + k\right)$ and so $P(k+1)$ is true.

Therefore, by the principle of mathematical induction, 3 is a factor of $n^3 - n + 3$ for all $n \geq 1$.

QUESTION 14

Use mathematical induction to prove that 9 is a factor of $10^{n+1}+3\cdot 10^{n}+5$ for all $n \geq 0$.

Solution

QUESTION 15

Use mathematical induction to prove that 4 is a factor of $5^n - 1$.

Solution

QUESTION 16

Use mathematical induction to prove that $x - y$ is a factor of $x^n - y^n$ **Hint:** Use the fact that $x^{n+1} - y^{n+1} = \frac{1}{2}\left[(x+y)\left(x^n - y^n\right) + (x-y)\left(x^n + y^n\right)\right]$.

Solution

SECTION 3: VECTORS

AREA OF STUDY 5: SPACE AND MEASUREMENT

VECTORS

(Vectors and Vector Calculus have been incorporated into the same Area of Study in the current study design. Many textbooks present these topics separately (as in previous study designs), some address these topics together. Your teacher at school may also choose to cover both topics later in the year or, like TSFX, cover Vectors in Unit 3 and Vector Calculus in Unit 4. Our Summer School program will therefore focus on Vectors. Vector Calculus will be addressed at Winter School and our other Unit 4 programs.)

VECTORS & SCALARS

Quantities that may be specified by simply stating their magnitude are referred to as **scalar** quantities.

For example:

- Distance
- Speed
- Mass

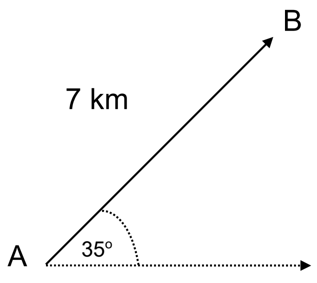
Quantities which need both magnitude and direction for their complete specification are known as **vector** quantities.

For example:

- Displacement 10 km NE
- Velocity 5 m/s, –5 m/s (These act in opposite directions.)

Vectors are geometrically represented using a directed line segment. The length of the line segment represents the magnitude of the vector, while the orientation shows the direction of the vector.

For example:

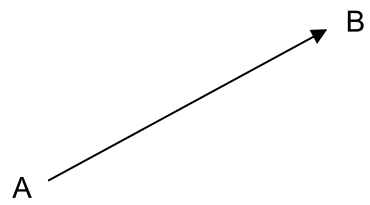


$\overrightarrow{AB}$ is a vector acting in the direction $N55°E$ with a magnitude of 7 km.

VECTOR NOTATION

Vectors are usually named using the small letters of the alphabet. These letters are written with a ~ (tilde) underneath. For example: $\underset{\sim}{b}$.

Vectors may also be named by stating the initial and terminal points (end points) using capital letters. For example, the below vector is represented as $\overrightarrow{AB}$.



The magnitude or modulus of the vector $\overrightarrow{AB}$ is denoted by $\left|\overrightarrow{AB}\right|$. In the same way, we may denote the magnitude of $\underset{\sim}{a}$ by $\left|\underset{\sim}{a}\right|$.

Vectors may be defined as being free or localised.

Localised vectors are characterised as having specific start and end points.

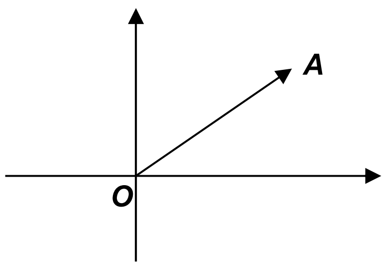
Free vectors are not localised i.e. they do not have specific start and end points in space.

POSITION VECTORS

When using Cartesian coordinate axes we refer to position vectors.

A position vector is a displacement vector whose starting point is the origin. Any point's position can be given in terms of a vector from the origin to that point.

For example:



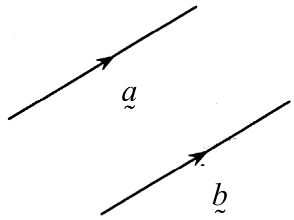
$\overrightarrow{OA}$ is the position vector of *A* relative to *O*.

Note:

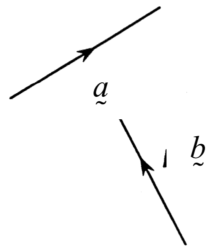
When a question says that the position vectors are given by $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$, this means that $\overrightarrow{OA} = \underset{\sim}{a},\ \overrightarrow{OB} = \underset{\sim}{b}\ \ and\ \ \overrightarrow{OC} = \underset{\sim}{c}$.

EQUALITY OF VECTORS

Two vectors are equal if they have the same magnitude and direction.

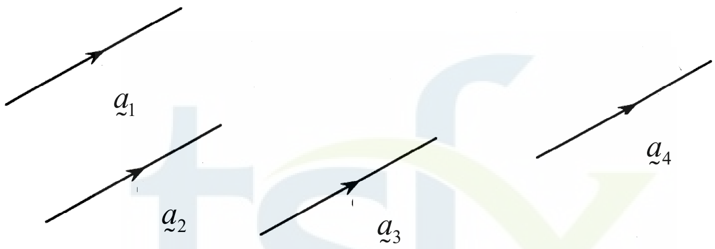


$\underset{\sim}{a} = \underset{\sim}{b}$

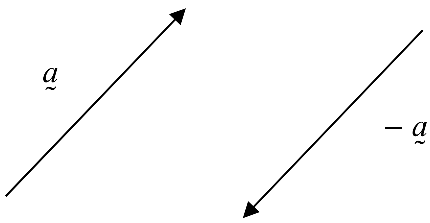


$\underset{\sim}{a} \neq \underset{\sim}{b}$

Note: Vectors that are equal do not need to have the same starting point. This means that vector $\underset{\sim}{a}$ can be represented by infinitely many directed line segments, each with the same magnitude and direction, as illustrated below.



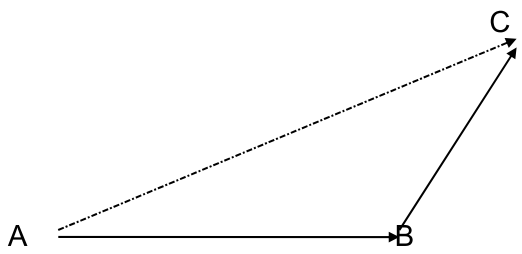
If $\underset{\sim}{a}$ is a vector then $-\underset{\sim}{a}$ is a vector with the same magnitude but acting in the opposite direction.



ADDITION OF VECTORS

When adding two vectors, you are finding the **resultant** vector.

For example:

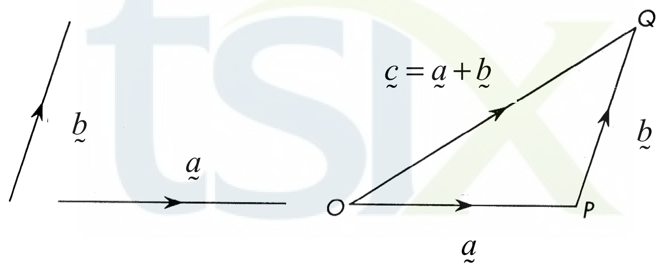


$\overrightarrow{AC}$ is the resultant vector.

Note: $\left|\overrightarrow{AC}\right| \leq \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$

The sum of two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, involves the combination not only of their magnitudes, but also of their directions.

To add two vectors, we use the "top to tail" method. This means that you draw the top of the first vector onto the tail of the second vector. The resultant is then drawn from the tail of the first vector to the top of the second vector.



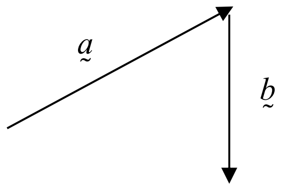
Note:

- $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| \leq \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$
- If $\left|\underset{\sim}{a} + \underset{\sim}{b}\right| = \left|\underset{\sim}{a}\right| + \left|\underset{\sim}{b}\right|$ then $\underset{\sim}{a}$ is parallel to $\underset{\sim}{b}$.
- Vectors obey the commutative law of addition i.e. $\underset{\sim}{a} + \underset{\sim}{b} = \underset{\sim}{b} + \underset{\sim}{a}$.
- Vectors obey the associative law of addition i.e. $\left(\underset{\sim}{a} + \underset{\sim}{b}\right) + \underset{\sim}{c} = \underset{\sim}{a} + \left(\underset{\sim}{b} + \underset{\sim}{c}\right)$.

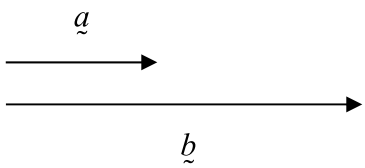
QUESTION 1

In each of the following cases, draw a directed line segment to represent vector $\underset{\sim}{c}$, where $\underset{\sim}{c} = \underset{\sim}{a} + \underset{\sim}{b}$.

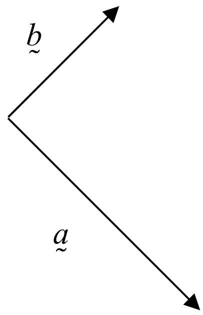
(a)



(b)



(c)



THE ZERO VECTOR

The zero vector is a vector which when added to any vector, leaves the vector unchanged.

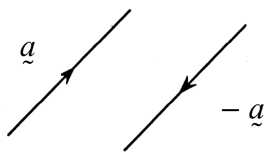
i.e. $\underset{\sim}{a} + \underset{\sim}{0} = \underset{\sim}{0} + \underset{\sim}{a} = \underset{\sim}{a}$

The zero vector has a zero magnitude and no particular direction.

THE INVERSE VECTOR

The inverse vector is one that when added to another vector, gives the zero vector.

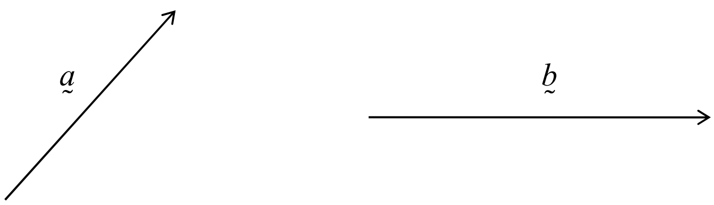
For each vector ($\underset{\sim}{a}$), there exists a unique vector $-\underset{\sim}{a}$, such that $\underset{\sim}{a} + -\underset{\sim}{a} = -\underset{\sim}{a} + \underset{\sim}{a} = \underset{\sim}{0}$.



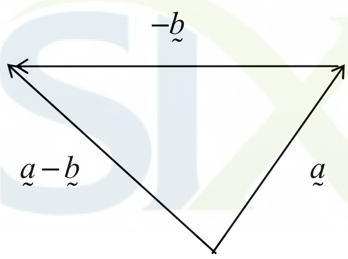
SUBTRACTION OF VECTORS

Subtraction is equivalent to the addition of a negative vector. Simply reverse the direction of the negative vector and add the two vectors "top to tail".

$$\underset{\sim}{a} - \underset{\sim}{b} = \underset{\sim}{a} + (-\underset{\sim}{b})$$



$\therefore\ \underset{\sim}{a} - \underset{\sim}{b} =$



QUESTION 2

Show that if the midpoints of each side of a rectangle are joined they form a parallelogram.

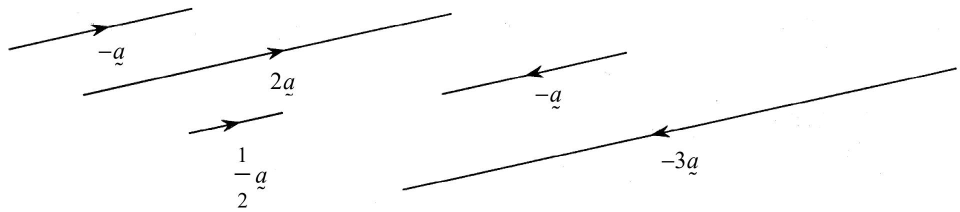
Solution

MULTIPLICATION OF A VECTOR BY A SCALAR

If $\underset{\sim}{a}$ is a vector and k is any real number, then $k\underset{\sim}{a}$ is a vector with a magnitude k times as big as $\underset{\sim}{a}$, acting parallel to $\underset{\sim}{a}$.

- If $k > 0$ then $k\underset{\sim}{a}$ has the same direction as $\underset{\sim}{a}$.
- If $k = 0$ then $k\underset{\sim}{a} = \underset{\sim}{0}$.
- If $k < 0$ then $k\underset{\sim}{a}$ has the opposite direction to $\underset{\sim}{a}$.

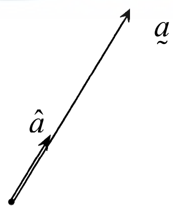
For example:



Note: Multiplication of a vector by a scalar results in a vector which is *k* times as long.
$k\underset{\sim}{a} = \underset{\sim}{a}k$

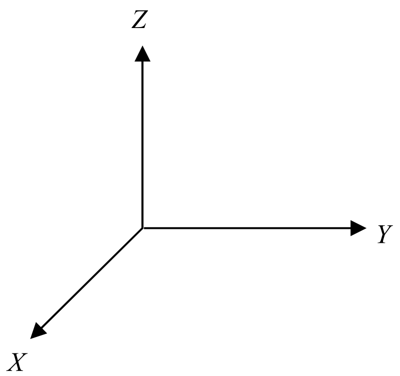
UNIT VECTORS

A **unit vector** is a vector whose magnitude is **one**, i.e. it is 1 unit long. These vectors are used as direction indicators, and are denoted by placing the symbol ^ on top of the vector name. For example, the unit vector in the direction of $\underset{\sim}{a}$ is denoted as $\underset{\sim}{\hat{a}}$.



VECTORS IN 3-DIMENSIONAL SPACE

To describe the position of a point in space, three coordinates are required. If we use the Cartesian system, only two of the coordinates of a point can be represented (x and y). The third Cartesian coordinate (z) is obtained by introducing a third axis, *OZ*, which lies perpendicular to the *X* and *Y* axis.



- A unit vector in the *X* direction is defined as $\underset{\sim}{i}$.
- A unit vector in the *Y* direction is defined as $\underset{\sim}{j}$.
- A unit vector in the *Z* direction is defined as $\underset{\sim}{k}$.

We can use this three-dimensional Cartesian system to describe the position of any point.

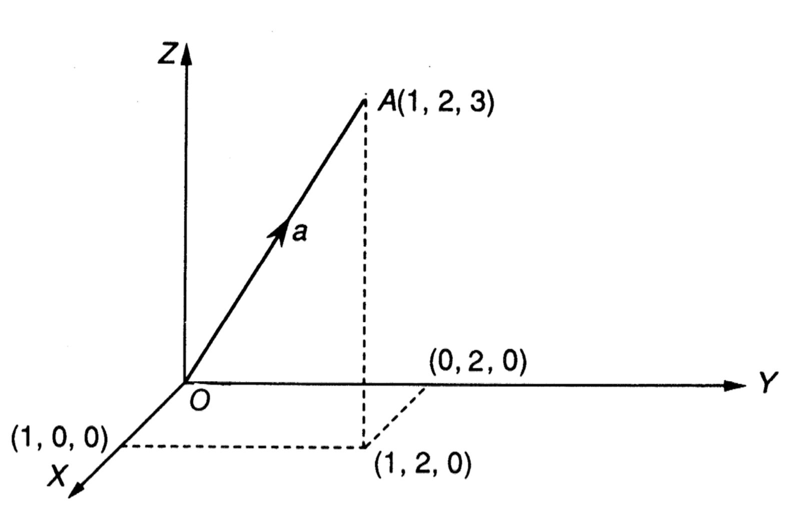
Note:

- To represent a point in vector notation we need to write its position in terms of $\underset{\sim}{i},\ \underset{\sim}{j}\ \text{ and }\ \underset{\sim}{k}.$

- The expression $3\underset{\sim}{i}$ represents a vector of magnitude 3 units parallel to the X-axis.

- The point ***P*** with co-ordinates (2, 1, 3) is written as $\overrightarrow{OP} = 2\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$.

  For example: $\overrightarrow{OA} = \underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$ is located 1 unit across in the *X* direction.
  2 units across in the *Y* direction.
  3 units up in the *Z* direction.



POSITION VECTORS

The vector $\overrightarrow{OP}$ is the position vector of ***P*** relative of *O*.

If a question states the position vector for ***A*** is $2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$ this really means that $\overrightarrow{OA}=2\underset{\sim}{i}+\underset{\sim}{j}+\underset{\sim}{k}$.

THE MAGNITUDE (LENGTH) OF A VECTOR

The magnitude (length) of a vector in three-dimensional space is determined by using an extension of Pythagoras' Theorem.

If $\overrightarrow{OP}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$ then the magnitude of $\overrightarrow{OP}$ is given by $\left|\overrightarrow{OP}\right|=\sqrt{x^2+y^2+z^2}$.
An alternative way of writing the vector is to use *p* tilde i.e. $\underset{\sim}{p}=x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}$

Simply:

Step 1: Square the coefficients of $\underset{\sim}{i}$, $\underset{\sim}{j}$, $\underset{\sim}{k}$.
Step 2: Add together.
Step 3: Square root the result.

For example:

If $\underset{\sim}{p}=\underset{\sim}{i}+2\underset{\sim}{j}+2\underset{\sim}{k}$ the magnitude of $\underset{\sim}{p}$ is equal to:

$$\left|\underset{\sim}{p}\right|=\sqrt{1^2+2^2+2^2}=\sqrt{9}=3$$

Vectors can be set up between any two points either in two or three dimensions.

For example:

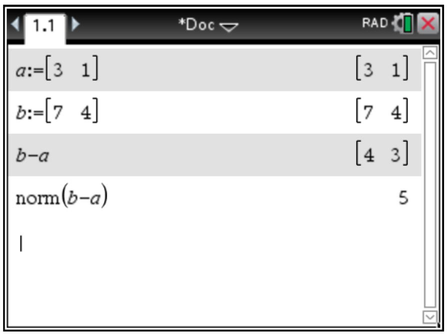
Find the distance between the points $A(3,1)$ and $B(7,4)$. The basic distance formula from Math Methods can be used where $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ or alternatively a vector $\overrightarrow{AB}$ between the two points can be created.

$\overrightarrow{OA}=3\underset{\sim}{i}+\underset{\sim}{j}$ and $\overrightarrow{OB}=7\underset{\sim}{i}+4\underset{\sim}{j}$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$$

$$\overrightarrow{AB}=4\underset{\sim}{i}+3\underset{\sim}{j}$$

$$\left|\overrightarrow{AB}\right|=\sqrt{4^2+3^2}=5$$



Therefore, the distance between the points is equal to 5 units.

The CAS calculator also numerous functions you should familiarize yourself with during the Vectors topic.

QUESTION 3
Find the distance between the points $P(1,0,-3)$ and $Q(2,5,-1)$.

Solution

CREATING UNIT VECTORS

To create a unit vector out of any vector, we simply divide the vector by its magnitude.

Thus, the unit vector of a vector $\underset{\sim}{a}$, in the direction of $\underset{\sim}{a}$ is $\hat{\underset{\sim}{a}} = \frac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

(Use small letter *p* with tilde throughout, not capital letter).

For example:

The magnitude of $\underset{\sim}{p}$ is 3. Therefore, the unit vector will be:

$$\hat{\underset{\sim}{p}} = \frac{1}{3}(\underset{\sim}{i} + 2\underset{\sim}{j} + 2\underset{\sim}{k})$$

This vector lies parallel to vector $\underset{\sim}{p}$ and has a magnitude of 1.