UNIT 3 MATHS METHODS

SUMMER SCHOOL

VCE SUMMER SCHOOL

Unit 3 Mathematical Methods

Area of Study 1

Functions, Relations & Graphs

Area of Study 2

Algebra, Number & Structure

Area of Study 3

Calculus

VCE Accreditation Period
2023 – 2027

www.tsfx.edu.au

TSFX

Suite 101 & 102
964 Mt Alexander Rd Essendon, VIC 3040

Phone: (03) 9663 3311
Email: admin@tsfx.edu.au

ABN: 70 066 955 128

VCE SUMMER SCHOOL HEAD START LECTURES

STUDY DESIGN (2023 – 2027) – EDITION 1

ACKNOWLEDGEMENTS

TSFX wishes to thank the Victorian Curriculum and Assessment Authority (VCAA) for permission to reproduce extracts from their publications.

ABOUT THE COVER IMAGE

THE POWER OF ART

Engaging with art is essential to the human experience. Almost as soon as motor skills are developed, children communicate through artistic expression. Throughout each stage of our lives, art plays different and important roles. The arts have the power to bring joy, stir up emotions and influence our behaviour. Art crosses all divides. It breaks down cultural, social and economic barriers and plays a big role in how humans see and interact with others, and the world in general.

Art decreases stress levels and improves mental health and well-being, particularly in patients suffering chronic or terminal illness. It has the power to educate people and convey meaning in a way that can be appreciated by every person. Furthermore, it gives us the opportunity to travel through time and learn from the beliefs, dreams, habits, thoughts, culture and lives of people in different places and times.

The arts also challenge us with different points of view, encourages communication, promotes stronger critical thinking and problem-solving skills and unlocks the potential of the human mind. It is also closely linked to academic achievement, civic engagement and social and emotional development.

The benefits of art are significant and undeniable. Use it to benefit both your mental and physical health as you journey through your VCE.

If you believe there is an error in this publication, please contact us as soon as possible (admin@tsfx.edu.au or 966 333 11). Amendments, errata, revisions, clarifications and further discussions will be published on our website at www.tsfx.edu.au/errata.

CONTENTS

UNIT 3 MATHEMATICAL METHODS

SECTION 1: UNIT 3 MATHEMATICAL METHODS COURSE OUTLINE & ASSESSMENT (2023 – 2027)

SECTION 2: AREA OF STUDY 2 ALGEBRA, NUMBER & STRUCTURE

SECTION 3: FUNCTIONS, RELATIONS AND GRAPHS

SECTION 4: SPECIAL FUNCTIONS AND RELATIONS

SECTION 5: CIRCULAR (TRIGONOMETRIC) FUNCTIONS

SECTION 6: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

SECTION 7: DIFFERENTIATION AND ITS APPLICATIONS

LOGARITHMS

The power in the expression $y = a^x$ is also known as the logarithm of y to the base a.
i.e. A logarithm is the power to which a number must be raised in order to get some other number.

Therefore, any indicial expression may be written in the corresponding logarithmic form by applying the rule:

If $\log_a y = x$ then $a^x = y,\quad a \in R^+ \setminus \{1\}$

$100 = 10^2$ reads as "2 is the logarithm of 100 to the base 10".

Note:

- a represents the base ($a \in R^+ \setminus \{1\}$).
- x represents the power/logarithm.
- y represents the basic numeral.

For Example: $3^4 = 81 \quad \Leftrightarrow \quad \log_3 81 = 4$

LOGARITHM LAWS

The following rules can only be applied if:

- Logarithms are written to the same base.
- There is no number in front of a logarithmic term.

Logarithm Law	Example
$\log_a m + \log_a n = \log_a (m \times n)$	$\log_2 5 + \log_2 3 = \log_2 (5 \times 3) = \log_2 (15)$
$\log_a m - \log_a n = \log_a \left(\frac{m}{n}\right)$	$\log_2 5 - \log_2 3 = \log_2 \left(\frac{5}{3}\right)$
$p \log_a (m) = \log_a (m)^p$	$3\log_2 5 = \log_2 5^3 = \log_2 125$
$\log_a \frac{1}{n} = \log_a 1 - \log_a n = -\log_a n$	$\log_3 9 = -\log_3 \left(\frac{1}{9}\right)$

Logarithm Law	Example
$\log_a\left(\frac{1}{a^x}\right)=-x$	$\log_8\left(\frac{1}{8^3}\right)=\log_8\left(8^{-3}\right)=-3\log_8\left(8\right)=-3$
$\log_a a=1$	$\log_6 6=1$
$\log_a a^x=x$	$\log_3 3^x=x$
$\log_a 1=0$	$\log_3 1=0$

SIMPLIFYING LOGARITHMIC EXPRESSIONS

Step 1: Write all terms as logarithmic expressions.

Step 2: Remove numbers in front of logarithmic expressions by converting them into powers.

Step 3: Apply the appropriate laws and simplify.

QUESTION 6
Simplify the following expressions stating your answer correct to 3 decimal places.

(a) $\log_{10} 20+\log_{10}\left(\frac{5}{4}\right)-\log_{10} 7=\log_{10}\left(20\times\frac{5}{4}\right)-\log_{10} 7$

$$=\log_{10} 25-\log_{10} 7$$

$$=\log_{10}\left(\frac{25}{7}\right)$$

(b) $3\log_3 6-2\log_3 18=\log_3 6^3-\log_3 18^2$

$$=\log_3\left(\frac{6^3}{18^2}\right)$$

$$=\log_3\left(\frac{216}{324}\right)$$

$$=\log_3\left(\frac{2}{3}\right)$$

(c) $3\log_3 2x + \log_3 2$

QUESTION 7

Write the following expressions as the sums and differences of their simplest forms.

(a) $\log_3\left(\dfrac{9x^4}{\sqrt{y}}\right)$

(b) $\log_3\left(\dfrac{(x+y)^2}{x^2+y^2}\right)$

SOLVING LOGARITHMIC EQUATIONS

Step 1: Write all terms as logarithmic expressions.

Convert numbers into logarithmic expressions by applying the rule: $\log_a a = 1$

Step 2: Remove numbers in front of logarithmic expressions by converting them into powers on basic numerals.

Step 3: Use logarithmic laws to reduce the given equation to as few terms as possible.

Step 4: Solve the equation and then verify that the calculated solutions are valid (substitute the calculated values into the **given equation**).

SOLUTION STRATEGIES

IF THE EQUATION CAN BE REDUCED TO 2 TERMS

(a) Write each term on either side of the equality sign and solve. You will typically obtain one of the following equation formats:

1. $\log_a \Delta = value$ **i.e. A logarithmic expression equal to a number.**

Write the logarithmic equation in its corresponding indicial form and then solve the expression using the techniques for solving indicial equations.

For example: Solve $\log_3 5 = x$

$$3^x = 5$$
$$\log_{10} 3^x = \log_{10} 5$$
$$x\log_{10} 3 = \log_{10} 5$$

$x = \dfrac{\log_{10} 5}{\log_{10} 3} = 1.465$ to 3 decimal places

2. $\log_a \Delta = \log_a \otimes$ **i.e. Two logarithmic expressions with the same base but different base numerals.**

Write each term on either side of the equality (one on each side of the equality sign) and equate base numerals.

For example: Solve $\log_6 2x = \log_6 (x+1)$

$$\log_6 2x = \log_6 (x+1)$$

$$\therefore 2x = x + 1$$
$$\therefore x = 1$$

3. $\log_a \Delta = \log_b \otimes$ **i.e. Two logarithmic expressions with different bases.**

Apply the change of base rule and solve.

For example: Solve $\log_5 x = \log_3 2$

4. $\log_a \Delta = function$ **i.e. A logarithmic expression equal to the equation a function other than a log eg. trigonometric, polynomial etc.**

These expressions must be solved using technology.

For example: $\log_{10}(6x-1) = \sin(\cos x)$

OTHER SOLUTION STRATEGIES

(a) If the given equation consists of three terms; two of which carry powers (where one power is double the other), write the given equation as a quadratic expression.

For example: $(\log_{10} x)^2 - 2\log_{10} x + 1 = 0$

(b) Bring all terms to one side of the equation, factorise and apply the Null Factor Law.

For example: $(\log_{10} x + 2)(\log_{10} x + 1) = 0$

CHANGE OF BASE RULE

A logarithmic expression whose base is not equal to e or 10 may be evaluated using technology, after applying the change of base rule.

To change the base from b to a we apply the rule: $\log_b x = \dfrac{\log_a x}{\log_a b}$.

Therefore, to change the base from b to 10 we apply the rule: $\log_b x = \dfrac{\log_{10} x}{\log_{10} b}$.

For example: $\log_2 7 = \dfrac{\log_{10} 7}{\log_{10} 2} = 2.81$ (to 2 dp)

IMPORTANT NOTES

- The logarithm of a zero or negative number is undefined. Therefore, after solving logarithmic equations, substitute solutions into the **original** equation to ensure that no solution obtained can render the logarithm undefined.
- Logarithmic functions of the form $y = \log_a(x-b)$ are only defined when $(x-b) > 0$.
- If asked to solve an inequality – proceed using an equality sign. Once solutions have been obtained, sketch a graph and use logic to determine the appropriate answer.
- **Note:** When an inequality is multiplied or divided by a negative number, the direction of the inequality must be changed. Remember that the logarithm of a number that lies between 0 and 1 is a negative.

QUESTION 8
Solve the following equations for x .

(a) $\log_x 256 = 4$

(b) $\log_2 16 = x - 1$

(c) $2\ln x = \ln(5+4x)$

(d) $\log_4(x-3)+\log_4 x = 1$

(e) $2(\log_{10} x)^2 - 7\log_{10} x = 4$

QUESTION 9
Simplify the following expressions.

(a) $(\log_3 9)(\log_4 1024)$

Let $\log_3 9 = x$ Let $\log_4 1024 = y$

$3^x = 9$ $4^y = 1024$

$3^x = 3^2$ $4^y = 4^5$

$x = 2$ $y = 5$ Answer is $2 \times 5 = 10$

(b) $\dfrac{\log_5 27}{\log_5 81}$

QUESTION 10
Consider the function $f:\{t : t < a\} \to R,\ f(t) = -5\log_e(4 - at)$, where a represents a positive real number value. The largest value of t for which $f(t)$ is defined is:

A 4

B $\dfrac{a}{4}$

C $-\dfrac{a}{4}$

D $\dfrac{4}{a}$

E $-\dfrac{4}{a}$

QUESTION 11

Given that $\log_a 5 + 2\log_a(2x+1) = \log_a 45$ then the value(s) of x which satisfy this equation are:

A 2
B 1, 2
C -2, 1
D 0, 1
E 1

QUESTION 12

Show that $3\log_3 6 = 2\log_3 18 + x$ can be written as $x = \log_3\left(\frac{2}{a}\right)$. Hence find the value of a.

QUESTION 13

If $2\log_m x = \log_m 9 + 8$ then x is equal to

A $\sqrt{17}$

B $\sqrt{17m}$

C $3m^4$

D 8.5

E $\pm 3m^4$

QUESTION 14

If $\log_e x = \log_e(x-1) + b$ then x is equal to

A $\dfrac{e^b}{1-e^b}$

B $\dfrac{1}{e^b - 1}$

C $\log_e \dfrac{x}{x-1}$

D $\dfrac{1}{1-e^b}$

E $\dfrac{e^b}{e^b - 1}$

QUESTION 15

$\log_2 63 = 3 - 2x$ can be written in the form $x = \dfrac{\log_e a}{2\log_e b} + \dfrac{3}{2}$. State the values of a and b.

QUESTION 16

Find the point(s) of intersection of the curves with equations $f(x) = \log_e e^{2z+4}$ and $g(x) = \log_e\left(\dfrac{8}{5}\right)$.

SOLVING INDICIAL EQUATIONS USING LOGARITHMS

Step 1: Apply the appropriate index laws to reduce both sides of the equation to one term.
Step 2: Take $\log_{10}$ or $\log_e$ of both sides of the equation.
Step 3: Solve for the required variable.

QUESTION 17

Solve $2^x = 0.5$ for x using logarithms.

Solution

$2^x = 0.5$

Take $\log_{10}$ of both sides: $\log_{10} 2^x = \log_{10} 0.5$

As $\log_a (m)^p = p \log_a (m)$

$$x \log_{10} 2 = \log_{10} 0.5$$
$$x = \frac{\log_{10} 0.5}{\log_{10} 2}$$
$$x = \frac{-0.301}{0.301}$$
$$x = -1$$

If no particular method was specified, the solution could be obtained using the CAS (Solve $2^x = 0.5$).

QUESTION 18

Solve $4^{x+2} = 14$.

QUESTION 19
Solve $e^{3x} = 2e^{x}$.

QUESTION 20
For what values of x is $0.5^{x} > 3$? State your answer correct to 3 decimal places.

QUESTION 21
Write $5e^{-3t} = 7e^{-8t}$ in its equivalent logarithmic form. Hence find a solution for t stating your answer correct to 4 decimal places.

Solution

$$\frac{5}{7} = \frac{e^{-8t}}{e^{-3t}} = e^{-8t-^{-}3t} = e^{-5t}$$

Take $\log_e$ of both sides of the equation:

$$\log_e\left(\frac{5}{7}\right) = \log_e e^{-5t}$$
$$\log_e\left(\frac{5}{7}\right) = -5t\log_e e$$

As $\log_e e = 1$:

$$\log_e\left(\frac{5}{7}\right) = -5t$$

$$t = -\frac{1}{5}\log_e\left(\frac{5}{7}\right) = 0.0673$$

If no particular method was specified, the solution could be obtained using the CAS (Solve $5e^{-3t} = 7e^{-8t}$).

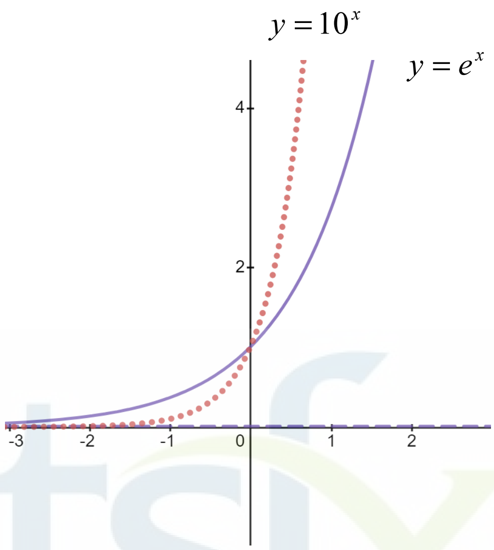
THE EXPONENTIAL GRAPH

General Equation: $y = a^x, \quad a > 1$

- The Y-intercept is $(0, 1)$.
- The horizontal asymptote is $y = 0$.
- The domain is R.
- The range is R^+.

The general shape of an exponential function depends upon the value of the base a.

When a > 1



As the value of the base "a" increases:

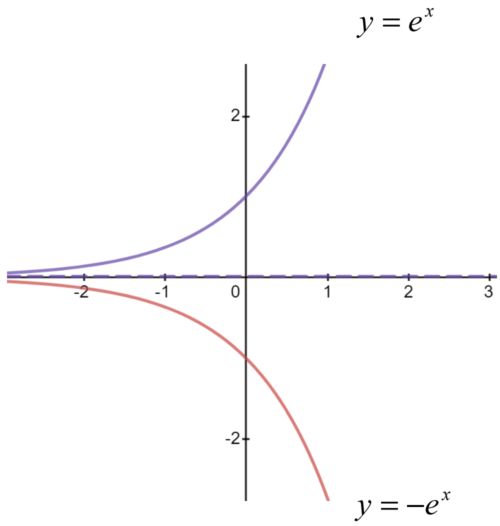
- The graph rises more steeply for positive values of x.
- The curve moves closer to the X-axis for negative values of x.

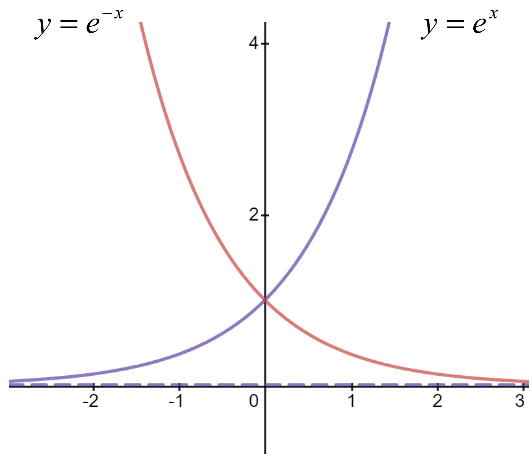
When 0 < a < 1: Graphs are obtained by reflecting curves with the reciprocal base in the Y-axis.

For example: $y = \left(\frac{1}{3}\right)^x = (3^{-1})^x = 3^{-x}$.

Therefore, the graph of $y = \left(\frac{1}{3}\right)^x$ is obtained by reflecting $y = 3^x$ in the Y-axis.

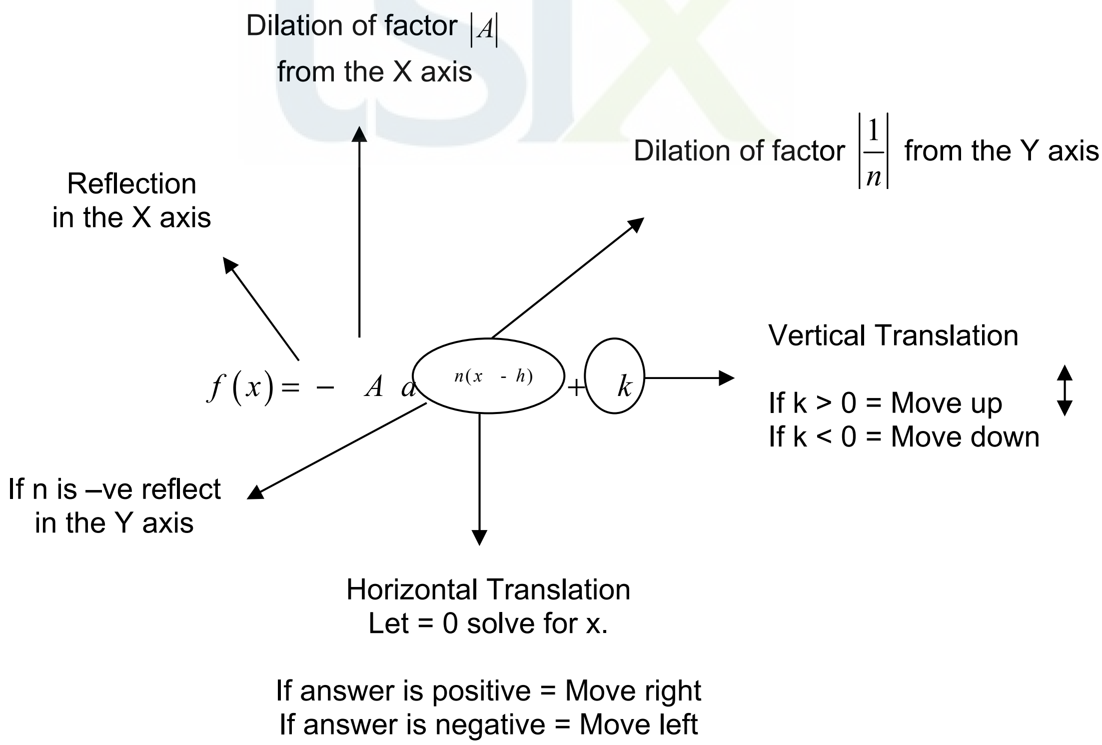
General Curve Shapes:





TRANSFORMATIONS INVOLVING EXPONENTIAL FUNCTIONS

Standard Form: $y = Aa^{n(x-h)} + k, \quad a > 0$



QUESTION 22

State the transformations of $y = e^x$ required to produce the graph of $y = 2e^{3-4x} - 5$.

Dilation from the Y axis **OR Parallel to the X axis** $f(x) \to f(nx)$ **Factor:**	**Dilation from the X axis** **OR Parallel to the Y axis** $f(x) \to af(x)$ **Factor:**
Reflection in the Y axis: $f(x) \to f(-x)$ **Yes** **No**	**Reflection in the X axis:** $f(x) \to -f(x)$ **Yes** **No**
Translation parallel to the X axis $f(x) \to f(x-h)$ **Number of Units:** **-ve direction** **+ve direction**	**Translation parallel to the Y axis** $f(x) \to f(x)+k$ **Number of Units:** **-ve direction** **+ve direction**

QUESTION 23

The function $f(x) = e^x$ is reflected in the X axis and dilated by a scale factor $\frac{1}{3}$ from the Y axis. The equation of the transformed curve is:

A $f(x) = -e^{(3x)}$

B $f(x) = e^{(-3x)}$

C $f(x) = -e^{(x/3)}$

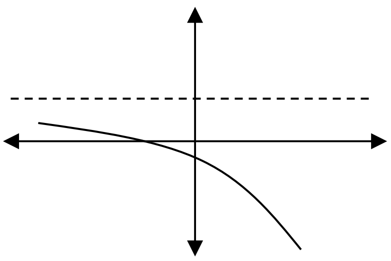
D $f(x) = e^{(-x/3)}$

E $f(x) = -e^{(-x/3)}$

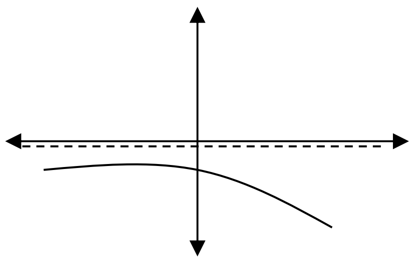
QUESTION 24

Which of the following best represents the graph of $y = ae^{kx} + c$, where $a > 0, \quad k > 0, \quad c < 0$?

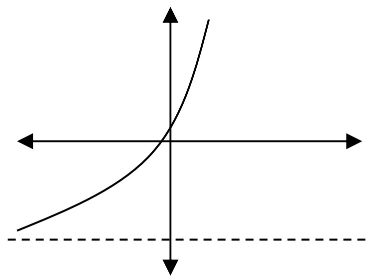
A



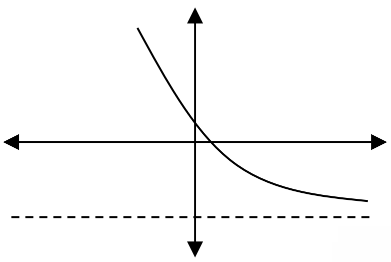
B



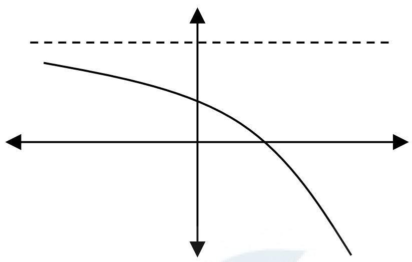
C



D



E



CAS Application

Let $a = 2, k = 3, c = -4$ and sketch $y = 2e^{3x} - 4$ on the calculator, The diagram that best fits the graph on the CAS is C.

QUESTION 25

Sketch the graph of $y = 2e^{3-2x} - 5$ stating the axial intercepts and the equation of the asymptote.

Solution

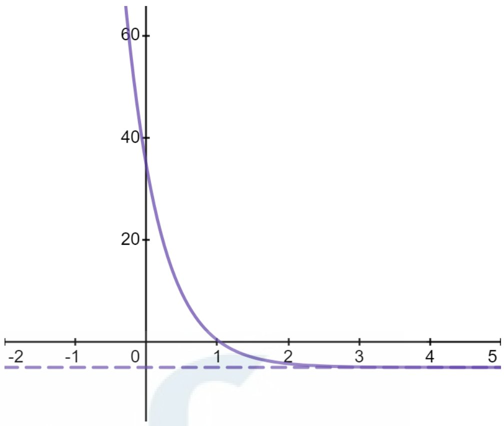
X intercepts – Let $y = 0$:

$$2e^{3-2x} - 5 = 0$$
$$2e^{3-2x} = 5$$
$$e^{3-2x} = \frac{5}{2}$$
$$\log_e e^{3-2x} = \log_e\left(\frac{5}{2}\right)$$
$$3 - 2x = \log_e\left(\frac{5}{2}\right)$$
$$x = \frac{3}{2} - \frac{1}{2}\log_e\left(\frac{5}{2}\right)$$

i.e. $\left(\frac{3}{2} - \frac{1}{2}\log_e\left(\frac{5}{2}\right), 0\right)$

Y intercepts – Let $x = 0$:

$$y = 2e^{3-2x} - 5$$
$$y = 2e^{3} - 5$$ i.e. $\left(0, 2e^{3} - 5\right)$



$y = -5$

QUESTION 26

Sketch the graph of $y = 2 \times 10^{2x} - 20$ showing the equations of any asymptotes as well as the axial intercepts.

QUESTION 27

Graph A is the graph of $y = a(3)^x$ and Graph B is the graph of $y = b(4)^x$ and $a, b > 0$ and $b > a$. Which of the following statements is correct?

A Both Graphs A and B rise at the same rate.
B Graph A rises at a faster rate than Graph B.
C Graph B rises at a faster rate than Graph A.
D The Y intercept of Graph A is above the Y intercept of Graph B when $b > a$.
E The equation of the X asymptote is dependent on the values of a and b.

FINDING EQUATIONS DESCRIBING EXPONENTIAL FUNCTIONS

$$y = Aa^{n(x-h)} + k, \quad a > 0$$

Step 1: Use features of the graph to solve for one unknown.

For Example: The asymptote represents the vertical translation (k).

Step 2: Substitute given points into the equation describing the curve and solve for the remaining unknowns.

Note:

- If the horizontal asymptote is located along the X axis, no vertical translation exists i.e. $k = 0$.

- The Y intercept is determined by the horizontal and vertical translations, together with the dilation from the X axis. Therefore, never use the Y intercept to determine the equation of the asymptote unless no dilations and horizontal translations exist.

QUESTION 28

The graph whose equation is $y = Ae^x + B$, where A and B are constants, is shown below. The values of A and B respectively are:

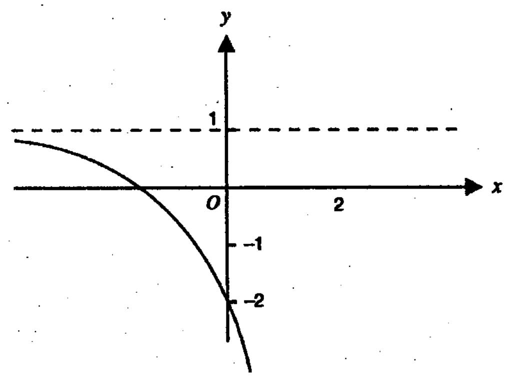
A $A=1$ $B=-2$

B $A=-2$ $B=1$

C $A=-1$ $B=-1$

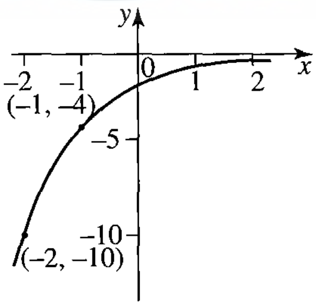
D $A=-3$ $B=1$

E $A=-1$ $B=-2$



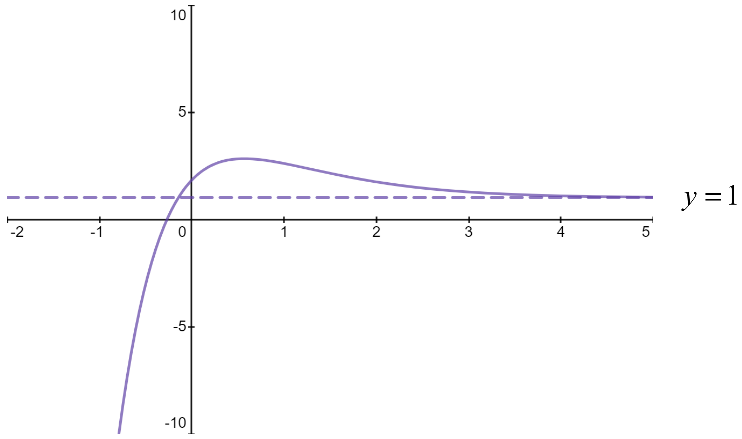
QUESTION 29

Find the values of A and k if the general equation of the graph shown is $y = Ae^{kx}$. Give your answer correct to 2 decimal places.



QUESTION 30

The graph of the function $f: R \to R,\ f(t) = (bt + 0.8)e^{-1.4t} + a$ is shown below.



This graph has an asymptote at $y = 1$ and passes through the point $(-3, -1086)$.

(a) Find the value of a.

(b) Show that b is equal to 5.70.

THE LOGARITHMIC GRAPH

The logarithmic curve is a reflection of the exponential graph in the line $y = x$.

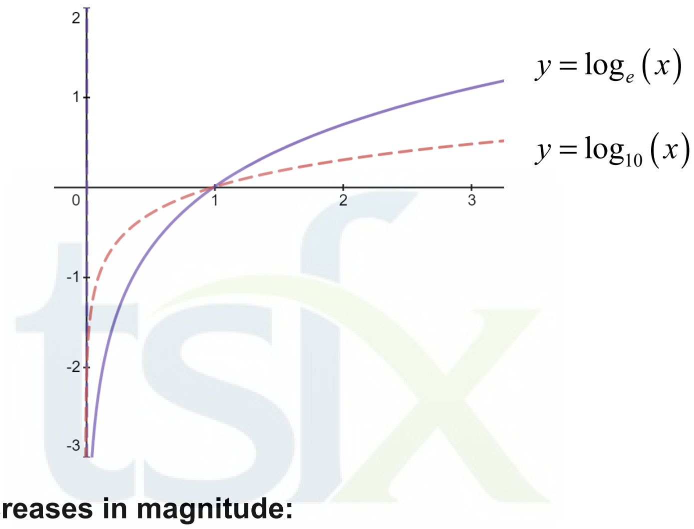
General Equation: $y = \log_a(x)$

- The X-intercept occurs at $(1, 0)$.
- The vertical asymptote is $x = 0$.
- The domain is R^+.
- The range is R.

Logarithmic graphs always have a vertical asymptote.

The general shape of a logarithmic function depends upon the value of the base a.

When a > 1



As the value of "a" increases in magnitude:

- The graph of $y = \log_a(x)$ rises more steeply for values of x less than one.
- The graph rises less significantly for values of x that are greater than one.

When 0 < a < 1:

These graphs are obtained by reflecting the graphs of logarithmic functions whose base is the reciprocal in the X-axis.

For example:

$$y = \log_{1/2}(x)$$

$$\therefore \left(\frac{1}{2}\right)^y = x$$

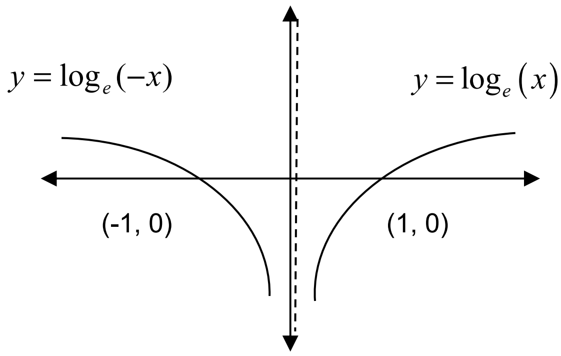
$$\left(2^{-1}\right)^y = x$$

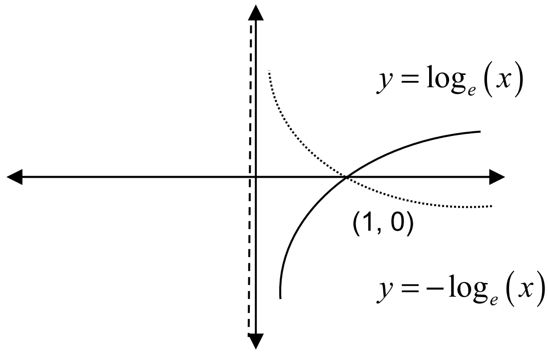
$$2^{-y} = x$$

$$-y = \log_2 x$$

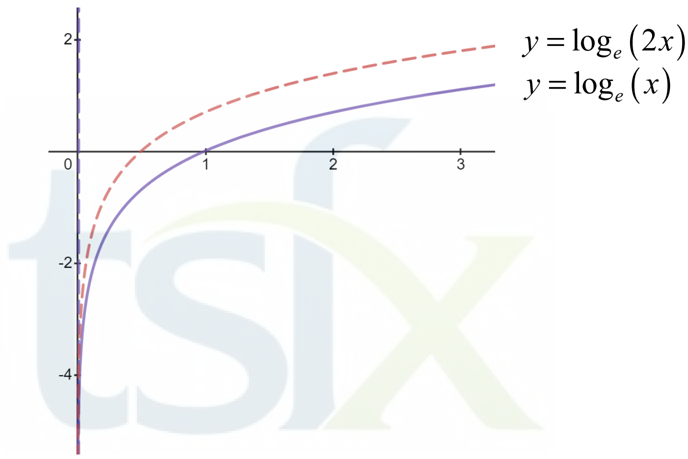
$y = -\log_2 x$ which is the reflection of $y = \log_{1/2}(x)$ in the X axis.

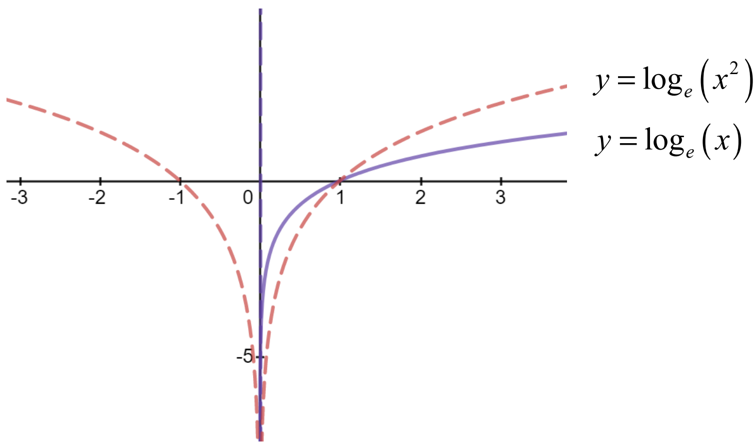
GENERAL CURVE SHAPES





SPECIAL GRAPHS





Note: $\log_e(x^2) = 2\log_e(x)$ only for $x > 0$