UNIT 1 MATHS METHODS

SUMMER SCHOOL

VCE SUMMER SCHOOL

Unit 1 Mathematical Methods

Area of Study 1

Functions, Relations & Graphs

Area of Study 2

Algebra, Number & Structure

Area of Study 3

Calculus

VCE Accreditation Period
2023 – 2027

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VCE SUMMER SCHOOL HEAD START LECTURES

STUDY DESIGN (2023 – 2027) – EDITION 1

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ABOUT THE COVER IMAGE
THE POWER OF ART

Engaging with art is essential to the human experience. Almost as soon as motor skills are developed, children communicate through artistic expression. Throughout each stage of our lives, art plays different and important roles. The arts have the power to bring joy, stir up emotions and influence our behaviour. Art crosses all divides. It breaks down cultural, social and economic barriers and plays a big role in how humans see and interact with others, and the world in general.

Art decreases stress levels and improves mental health and well-being, particularly in patients suffering chronic or terminal illness. It has the power to educate people and convey meaning in a way that can be appreciated by every person. Furthermore, it gives us the opportunity to travel through time and learn from the beliefs, dreams, habits, thoughts, culture and lives of people in different places and times.

The arts also challenge us with different points of view, encourages communication, promotes stronger critical thinking and problem-solving skills and unlocks the potential of the human mind. It is also closely linked to academic achievement, civic engagement and social and emotional development.

The benefits of art are significant and undeniable. Use it to benefit both your mental and physical health as you journey through your VCE.

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SECTION 1: UNIT 1 MATHEMATICAL METHODS COURSE OUTLINE & ASSESSMENT (2023 – 2027)

VCE STUDY DESIGN
ACCREDITATION PERIOD: 2023 – 2027

AREA OF STUDY 1
FUNCTIONS, RELATIONS & GRAPHS

In this area of study students cover the graphical representation of simple algebraic functions (polynomial and power functions) of a single real variable and the key features of functions and their graphs such as axis intercepts, domain (including the concept of maximal, natural or implied domain), co-domain and range, stationary points, asymptotic behaviour and symmetry. The behaviour of functions and their graphs is to be explored in a variety of modelling contexts and theoretical investigations.

This Area of Study Includes:

- Functions and function notation, domain, co-domain and range, representation of a function by rule, graph and table, inverse functions and their graphs.
- Qualitative interpretation of features of graphs of functions, including those of real data not explicitly represented by a rule, with approximate location of any intercepts, stationary points and points of inflection.
- Graphs of power functions $f(x) = x^n$ where $n \in \left\{-2, -1, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4\right\}$ and transformations of these graphs to the form $y = a(x+b)^n + c$ where $a, b, c \in R$ and $a \neq 0$.
- Graphs of polynomial functions of low degree, and interpretation of key features of these graphs.

AREA OF STUDY 2
ALGEBRA, NUMBER AND STRUCTURE

This area of study supports students' work in the 'Functions, relations and graphs', 'Calculus' and 'Data analysis, probability and statistics' areas of study, and content is to be distributed between Units 1 and 2.

In Unit 1 the focus is on the algebra of polynomial functions of low degree and transformations of the plane.

This Area of Study Includes:

- Use of symbolic notation to develop algebraic expressions and represent functions, relations, equations, and systems of simultaneous equations.
- Substitution into, and manipulation of, these expressions.
- Recognition of equivalent expressions and simplification of algebraic expressions involving different forms of polynomial and power functions, the use of distributive and exponent laws applied to these functions, and manipulation from one form of expression to an equivalent form.
- Use of parameters to represent families of functions and determination of rules of simple functions and relations from given information.

- Transformations of the plane and application to basic functions and relations by simple combinations of dilations (students should be familiar with both ‘parallel to an axis’ and ‘from an axis’ descriptions), reflections in an axis and translations (matrix representation may be used but is not required).

- The connection between the roots of a polynomial function, its factors and the horizontal axis intercepts of its graph, including the remainder, factor and rational root theorems.

- Solution of polynomial equations of low degree, numerically, graphically and algebraically, including numerical approximation of roots of simple polynomial functions using the bisection method algorithm.

- Solution of a set of simultaneous linear equations (geometric interpretation only required for two variables) and equations of the form $f(x)=g(x)$ numerically, graphically and algebraically.

AREA OF STUDY 3
CALCULUS

In this area of study students cover constant and average rates of change and an introduction to instantaneous rate of change of a function in familiar contexts, including graphical and numerical approaches to estimating and approximating these rates of change.

This Area of Study Includes:

- Average and instantaneous rates of change in a variety of practical contexts and informal treatment of instantaneous rate of change as a limiting case of the average rate of change.

- Interpretation of graphs of empirical data with respect to rate of change such as temperature or pollution levels over time, motion graphs and the height of water in containers of different shapes that are being filled at a constant rate, with informal consideration of continuity and smoothness.

- Use of gradient of a tangent at a point on the graph of a function to describe and measure instantaneous rate of change of the function, including consideration of where the rate of change is positive, negative or zero, and the relationship of the gradient function to features of the graph of the original function.

AREA OF STUDY 4
DATA ANALYSIS, PROBABILITY AND STATISTICS

In this area of study students cover the concepts of experiment (trial), outcome, event, frequency, probability and representation of finite sample spaces and events using various forms such as lists, grids, Venn diagrams and tables. They also cover introductory counting principles and techniques and their application to probability.

This Area of Study Includes:

- Random experiments, sample spaces, outcomes, elementary and compound events, random variables and the distribution of results of experiments.
- Simulation using simple random generators such as coins, dice, spinners and pseudo-random generators using technology, and the display and interpretation of results, including informal consideration of proportions in samples.
- Addition and multiplication principles for counting.
- Combinations including the concept of a selection and computation of ${}^{n}C_{r}$ and the application of counting techniques to probability.

MATHEMATICAL INVESTIGATION

This comprises one to two weeks of investigation into one or two practical or theoretical contexts or scenarios based on content from areas of study and application of key knowledge and key skills for the outcomes.

Investigation is to be incorporated in the development of concepts, skills and processes for the unit, and can be used to assess the outcomes.

There are three components to mathematical investigation:

Formulation:

Overview of the context or scenario, and related background, including historical or contemporary background as applicable, and the mathematisation of questions, conjectures, hypotheses, issues or problems of interest.

Exploration:

Investigation and analysis of the context or scenario with respect to the questions of interest, conjectures or hypotheses, using mathematical concepts, skills and processes, including the use of technology and application of computational thinking.

Communication:

Summary, presentation and interpretation of the findings from the mathematical investigation and related applications.

OUTCOMES

For this unit the student is required to demonstrate achievement of three outcomes. As a set these outcomes encompass all of the areas of study for the unit.

OUTCOME 1

On completion of this unit the student should be able to define and explain key concepts as specified in the content from the areas of study and apply a range of related mathematical routines and procedures.

To achieve this outcome the student will draw on key knowledge and key skills outlined in all the areas of study.

OUTCOME 1: KEY KNOWLEDGE

- The definition of a function, the concepts of domain (including maximal, natural or implied domain), co-domain and range, notations for specification of the domain, co-domain and range and rule of a function.
- The key features and properties of power and polynomial functions and their graphs, including any vertical or horizontal asymptotes.
- The effect of transformations of the plane, dilation, reflection in axes, translation and simple combinations of these transformations, on the graphs of functions.
- The relation between the graph of a one-to-one function, its inverse function and reflection in the line $y = x$.
- Representations of points and transformations.
- Factorisation patterns, the quadratic formula and discriminant, the remainder, factor and rational root theorems and the null factor law.
- The exponent laws.
- Average and instantaneous rates of change and their interpretation with respect to the graphs of functions.
- Forms of representation of sample spaces and events.
- The properties that probabilities for a given sample space are non-negative and the sum of these probabilities is one.
- Counting techniques and their application to probability.

OUTCOME 1: KEY SKILLS

- Specify the rule, domain (including maximal, natural or implied domain), co-domain, and range of a function and identify whether or not a relation is a function.
- Substitute integer, simple rational and irrational numbers in exact form into expressions, including rules of functions and relations, and evaluate these by hand.
- Re-arrange and solve simple algebraic equations and inequalities by hand.
- Expand and factorise linear and simple quadratic expressions with integer coefficients by hand.
- Express $ax^2 + bx + c$ in completed square form where $a, b, c \in Z$ and $a \neq 0$, by hand.
- Express a cubic polynomial $p(x)$, with integer coefficients, in the form $p(x) = (x - a)q(x) + r$ and determine $\frac{p(x)}{x - a}$ by hand.
- Use algebraic, graphical and numerical approaches, including the factor theorem and the bisection method algorithm, to determine and verify solutions to equations over a specified interval.
- Apply distributive and exponent laws to manipulate and simplify expressions involving polynomial and power function, by hand in simple cases.
- Set up and solve systems of simultaneous linear equations involving up to four unknowns, including by hand for a system of two equations in two unknowns.
- Sketch by hand graphs of linear, quadratic and cubic polynomial functions, and quartic polynomial functions in factored form (approximate location of stationary points only for cubic and quartic functions), including cases where an x-axis intercept is a touch point or a stationary point of inflection.
- Sketch by hand graphs of power functions $f(x) = x^n$ where $n \in \left\{-2, -1, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4\right\}$ and simple transformations of these, and identify any vertical or horizontal asymptotes.
- Draw graphs of polynomial functions of low degree, simple power functions and simple relations that are not functions.
- Describe the effect of transformations on the graphs of relations and functions.
- Sketch the graph of the inverse function of a one-to-one function given its graph.
- Use graphical, numerical and algebraic approaches to find an approximate value or the exact value (as appropriate) for the gradient of a secant or tangent to a curve at a given point.
- Set up probability simulations, and describe the notions of randomness and variability, and their relation to events.
- Apply counting techniques to solve probability problems and calculate probabilities for compound events, by hand in simple cases.

OUTCOME 2

On completion of this unit the student should be able to apply mathematical processes in non-routine contexts, including situations with some open-ended aspects requiring investigative, modelling or problem-solving techniques or approaches, and analyse and discuss these applications of mathematics.

To achieve this outcome the student will draw on key knowledge and key skills outlined in all the areas of study.

OUTCOME 2: KEY KNOWLEDGE

- Key mathematical content from one or more areas of study related to a given context.
- Specific and general formulations of concepts used to derive results for analysis within a given context.
- The role of examples, counter-examples and general cases in working mathematically.
- Key elements of algorithm design: sequencing, decision-making, repetition and representation including the use of pseudocode.
- Inferences from analysis and their use to draw valid conclusions related to a given context.

OUTCOME 2: KEY SKILLS

- Specify the relevance of key mathematical content from one or more areas of study to the investigation of various questions in a given context.
- Identify important information, variables, constraints and other key features to the investigation of various questions in a given context.
- Develop mathematical formulations of specific and general cases used to derive results for analysis within a given context.
- Use a variety of techniques to verify results.
- Make inferences from analysis and use these to draw valid conclusions related to a given context.
- Communicate results and conclusions using both mathematical expression and everyday language, in particular, the interpretation of mathematics with respect to a context.

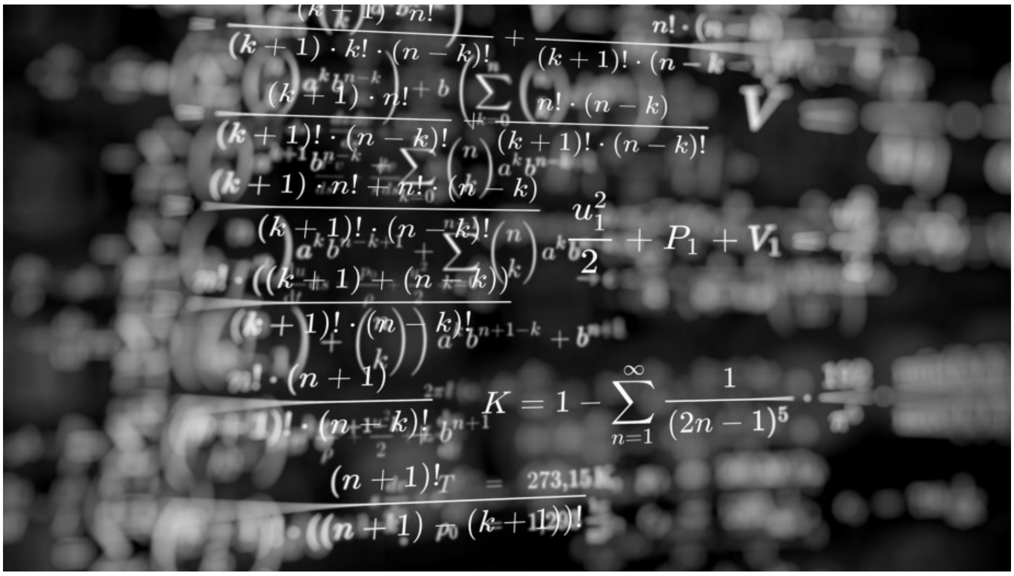
OUTCOME 3

On completion of this unit the student should be able to apply computational thinking and use numerical, graphical, symbolic and statistical functionalities of technology to develop mathematical ideas, produce results and carry out analysis in situations requiring investigative, modelling or problem-solving techniques or approaches.

To achieve this outcome the student will draw on key knowledge and key skills outlined in all the areas of study.

OUTCOME 3: KEY KNOWLEDGE

- The role of computational thinking (abstraction, decomposition, pattern and algorithm) in problem-solving, and its application to mathematical investigation.
- Exact and approximate specification of mathematical information such as numerical data, graphical forms and general or specific forms of solutions of equations produced by use of technology.
- Domain and range requirements for specification of graphs of functions and relations when using technology.
- The role of parameters in specifying general forms of functions and equations.
- The relation between numerical, graphical and symbolic forms of information about functions and equations and the corresponding features of those functions and equations.
- The similarities and differences between formal mathematical expressions and their representation by technology.
- The purpose and effect of sequencing, decision-making and repetition statements on relevant functionalities of technology, and their role in the design of algorithms and simulations.
- The appropriate functionality of technology for a variety of mathematical contexts.



OUTCOME 3: KEY SKILLS

- Use computational thinking, algorithms, models and simulations to solve problems related to a given context.
- Distinguish between exact and approximate presentations of mathematical results produced by technology, and interpret these results to a specified degree of accuracy.
- Use technology to carry out numerical, graphical and symbolic computation as applicable.
- Produce results, using technology, which identify examples or counter-examples for propositions.
- Produce tables of values, families of graphs and collections of other results using technology, which support general analysis in investigative, modelling and problem-solving contexts.
- Use appropriate domain and range specifications to illustrate key features of graphs of functions and relations.
- Identify the relation between numerical, graphical and symbolic forms of information about functions and equations, and the corresponding features of those functions and equations.
- Specify the similarities and differences between formal mathematical expressions and their representation by technology, in particular, equivalent forms of symbolic expressions.
- Select an appropriate functionality of technology in a variety of mathematical contexts and provide a rationale for these selections.
- Design and implement simulations and algorithms using appropriate functionalities of technology.
- Apply suitable constraints and conditions, as applicable, to carry out required computations.
- Relate the results from a particular technology application to the nature of a particular mathematical task (investigative, modelling or problem-solving) and verify these results.
- Specify the process used to develop a solution to a problem using technology and communicate the key stages of mathematical reasoning (formulation, solution, interpretation) used in this process.

ASSESSMENT

The award of satisfactory completion for a unit is based on whether the student has demonstrated achievement of the set of outcomes specified for the unit. Teachers should use a variety of learning activities and assessment tasks that provide a range of opportunities for students to demonstrate the key knowledge and key skills in the outcomes.

The areas of study and the key knowledge and key skills listed for the outcomes, should be used for course design and the development of learning activities and assessment tasks.

Assessment must be a part of the regular teaching and learning program and should be completed mainly in class and within a limited timeframe.

All assessments at Units 1 and 2 are school-based. Procedures for assessment of levels of achievement in Units 1 and 2 are a matter for school decision.

For this unit students are required to demonstrate achievement of three outcomes. As a set these outcomes encompass the areas of study in the unit.

Suitable tasks for assessment in this unit may be selected from the following:

Demonstration of achievement of Outcome 1 should be based on the student's performance on a selection of the following assessment tasks:

- Assignments
- Tests
- Solutions to sets of worked questions
- Summary notes or review notes

Demonstration of achievement of Outcome 2 should be based on the student's performance on mathematical investigations and a selection of modelling or problem-solving tasks.

Demonstration of achievement of Outcome 3 should be based on the student’s performance on aspects of tasks completed in demonstrating achievement of Outcomes 1 and 2 that incorporate opportunity for computational thinking and the effective and appropriate use of technology.

Where teachers allow students to choose between tasks, they must ensure that the tasks they set are of comparable scope and demand.

SECTION 2: AREA OF STUDY 2
ALGEBRA, NUMBER & STRUCTURE

TOPIC 1: ALGEBRA

SUBSTITUTION OF VALUES

Mathematical formulae are commonly used in many areas of life. We are often required to substitute values of pronumerals (letters) into formulae so as to find the value of an unknown pronumeral. This process is referred to as **substitution**.

Note:

You must be very careful when substituting values into the calculator. The order of operations (BIDMAS or BODMAS) must always be obeyed and brackets should be used where possible.

QUESTION 1

(a) Find the value of P when $l = 24$ and $w = 10$, if $P = 2(l + w)$.

Substituting the value of l and w into the given formula gives:

$$\begin{aligned} P &= 2(24 + 10) \\ &= 2 \times 34 \\ &= 68\ units \end{aligned}$$

This particular formula relates to the perimeter, P, of a rectangle, whose length, $l = 24$ and width, $w = 10$. This is one of the many applications of a mathematical formula.

(b) Find the value of A when $r = 4$ if $A = \pi r^2$.

The value of A is found by substituting the given value of r into the formula.

$$\begin{aligned} A &= \pi \times (4)^2 \\ &= \pi \times 16 \\ &= 16\pi \\ &= 50.265\ units^2 \end{aligned}$$

This example relates to the area of a circle whose radius is 4.

(c) Find s when $u = 0,\ a = 10,\ t = 4$ if $s = ut + \frac{1}{2}at^2$.

The required value of s is found by substituting the given values of $u,\ a$ and t into the given formula.

$$s = (0\times 4) + \frac{1}{2}\times 10\times (4)^2$$
$$= 0 + \frac{1}{2}\times 10\times 16 \ = 80\,units$$

This is a well known formula in physics that is used to find the displacement travelled by a body, s, given the initial velocity, u, the acceleration, a, and the time of travel, t.

(d) Find A when $P = 1000$, $r = 10$ and $n = 5$, if $A = P\left(1 + \frac{r}{100}\right)^n$.

This example relates to finding the amount of money, *P*($), invested at a compound interest rate of 10% for a period of 5 years.

REARRANGEMENT AND SUBSTITUTION

Rearrangement and substitution involves finding the value of a pronumeral which is not initially the subject. In such examples, it is necessary to first rearrange the formula (to make the required pronumeral the subject), before substituting the given values.

To rearrange the given formula, we "undo" each mathematical operation one step at a time.

- Addition will undo subtraction.
- Subtraction will undo addition.
- Multiplication will undo division.
- Division will undo multiplication.
- Squaring will undo a square root.
- The square root will undo a squared term.

Remember to perform the same operation to BOTH sides of the equation.

Order of Operation: BIDMAS

- Brackets (from left to right)
- Index
- Division (from left to right)
- Multiplication (from left to right)
- Addition (from left to right)
- Subtraction (from left to right)

QUESTION 2

(a) If $P = 2(l + w)$, find the value of l if $P = 48$ and $w = 16$.

Rearrange the equation to make l the subject:

$P = 2(l + w)$

$= 2l + 2w$ Expand brackets

$2l = P - 2w$ Make $2l$ the subject

$l = \dfrac{P - 2w}{2}$ Make l the subject

Substitute the given data and solve for the value of l:

Substitute $P = 48,\ W = 16$

$$\therefore l = \frac{48 - (2 \times 16)}{2} = \frac{48 - 32}{2} = 8\ units$$

(b) Find the radius, r, of a circle whose area, A, is $45\ cm^2$, given that $A = \pi r^2$.

Rearrange the equation to make r the subject:

$A = \pi r^2$ Make r^2 the subject by dividing both sides of the equation by π.

$r^2 = \dfrac{A}{\pi}$ Make r the subject by taking the square root of both sides.

$$r = \sqrt{\frac{A}{\pi}}$$

Substitute the given data and solve for the value of r:

$$r = \sqrt{\frac{45}{\pi}} = 3.785\ cm$$

(c) Find the value of a if $s = 25$, $u = 2$ and $t = 2$, given that $s = ut + \frac{1}{2}at^2$.

Rearrange the equation to make a the subject:

$$s = ut + \frac{1}{2}at^2$$

$\frac{1}{2}at^2 = s - ut$ Make $\frac{1}{2}at^2$ the subject by subtracting *ut* from both sides.

$at^2 = 2(s - ut)$ Multiply by 2 to eliminate the fraction.

$a = \dfrac{2(s - ut)}{t^2}$ Divide both sides by t^2 to make a the subject.

Substitute the given data and solve for the value of a:

$$a = \frac{2(25 - (2 \times 2))}{2^2} = \frac{2(21)}{4} = \frac{42}{4} = 10.5\ units$$

(d) Find the value of r when $A = 2000$, $P = 1000$ and $n = 5$, given that $A = P\left(1 + \frac{r}{100}\right)^n$.

QUESTION 3

(a) Solve $\frac{2}{n} = \frac{5}{n-3}$ for n.

(b) Solve $\frac{4}{d+3}=\frac{6}{d-1}$ for d.

QUESTION 4

Solve $I=\frac{PN}{RN+A}$ for N.

Solution

EXPANDING EXPRESSIONS

Expanding involves the removal of brackets from expressions, and is the opposite process to factorisation (insertion of brackets into expressions).

We use the distributive law to remove brackets from an equation.

THE DISTRIBUTIVE LAW

When an equation of the form $a(b+c)$ is expanded, every term inside the bracket is multiplied by the number or pronumeral (letter), and the sign that is located outside the brackets. This rule is known as the **Distributive Law**.

$$a(b+c) = (a \times b) + (a \times c) = ab + ac$$

Note: To avoid mistakes, include arrows above or below the terms that are being multiplied.

To expand a quadratic expression we often use **"FOIL"** i.e. First, Outside, Inside, Last.

$$(a+b)(c+d) = ac + ad + bc + bd$$

The distributive law also applies to equations that contain more than two terms inside the brackets:

$$a(b+c+d) = ab + ac + ad \qquad a(b+c+d+e) = ab + ac + ad + ae$$

When expanding expressions, we remove brackets, and then simplify by collecting like terms.

There is usually some simplifying to do afterwards which includes collecting like terms. This process can be extended to expand three binomial factors i.e. cubic expansions.

For Example:

Expand $(x+1)(x+3)(4-x)$. ***Do not*** attempt to expand all three brackets at one time

$$(x+1)(x+3)(4-x) = (x^2+3x+x+3)(4-x)$$ * Expand the first two brackets

$$= (x^2+4x+3)(4-x)$$
$$= (4x^2 - x^3 + 16x - 4x^2 + 12 - 3x)$$
$$= -x^3 + 13x + 12$$

QUESTION 5

Expand the following expressions.

(a) $\left(-3y^4\right)\left(2y^2\right)$

$\left(-3y^4\right)\left(2y^2\right)=\left(-3\times2\right)\left(y^4.y^2\right)=-6y^{4+2}=-6y^6$

(b) $\left(3xy^5\right)\left(-6x^4y^2\right)$

(c) $2x^3(-3x^4+2x^3-10x^2+7x+9)$

$$2x^3(-3x^4+2x^3-10x^2+7x+9)$$
$$=(2x^3)(-3x^4)+(2x^3)(2x^3)+(2x^3)(-10x^2)+(2x^3)(7x)+(2x^3)(9)$$
$$=-6x^7+4x^6-20x^5+14x^4+18x^3$$

(d) $-7a^2bc^3(5a^2-3b^2-9c^2)$

(e) $(3x^2+2x-5)(2x-3)$

(f) $(x^2-9)(4x^4+5x^2-2)$

EXPANDING EXPRESSIONS BY RULE

Some quadratic and cubic expressions will always follow a certain pattern, and therefore, we can expand these expressions by using a standard set of rules.

PERFECT SQUARES

Perfect squares are expressions that are written to the power of two.

For example: x^2, $(x+3)^2$ and $(x^3-x+1)^2$.

Perfect squares may be expanded directly or by applying the following rules:

$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$

METHOD:

Step 1: Let a represent the first term.
Let b represent the second term.

Step 2: Substitute the expressions for a and b into the appropriate rule.

Take care to avoid confusing the following:

$$(a+b)^2 \neq a^2 + b^2$$
$$(a-b)^2 \neq a^2 - b^2$$

QUESTION 6

Expand and simplify each of the following expressions:

(a) $(3a-b)^2 = (3a)^2 - 2(3a)(b) + (b)^2$

$= 9a^2 - 6ab + b^2$

(b) $(5x+7y)^2$

(c) $(1+\sqrt{x})^2$

(d) $[x^2-(3x+1)]^2$

THE DIFFERENCE OF TWO SQUARES (DOTS)

Given the product of two expressions that consist of the sum and difference of the same terms, we expand by applying the Difference of Two Squares in reverse.

$$(a+b)(a-b)=a^2-b^2$$

METHOD:

Step 1: Let a represent the first term.

Step 2: Let b represent the second term.

Step 3: Substitute the expressions for a and b into a^2-b^2.

QUESTION 7
Expand and simplify the following expressions:

(a) $(7x+2)(7x-2)=(7x)^2-(2)^2$

$$=49x^2-4$$

(b) $(\sqrt{x}-5x)(\sqrt{x}+5x)$

(c) $3(5a-b+2)(5a-b-2)$

EXPANDING PERFECT CUBES BY RULE

Perfect cubes are expressions that are written to the power of three.

For example: x^3, $(py)^3$, $(x+3)^3$, $(ax+6)^3$.

Perfect cubes may be expanded directly or by applying the following rules:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

METHOD:

Step 1: Let a represent the first term.
Let b represent the second term.

Step 2: Substitute the expressions for a and b into the appropriate rule.

Take care to avoid confusing the following:

$$(a+b)^3 \neq a^3 + b^3$$
$$(a-b)^3 \neq a^3 - b^3$$

QUESTION 8
Expand the following expressions:

(a) $(2x+3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)(3)^2 + (3)^3$

$= 8x^3 + 3(4x^2)(3) + 3(2x)(9) + 27$
$= 8x^3 + 36x^2 + 54x + 27$

(b) $(3x-8y)^3$

TECHNIQUES IN FACTORISATION

The process where brackets are inserted into an equation is referred to as **factorisation**. Factorisation is the opposite process to expansion.

Expansion

$$(x+3)(x-5) \quad \rightarrow \quad x^2-2x-15$$
$$\leftarrow$$

Factorisation

METHOD:

Bring all terms to one side of the equation and simplify by applying one or more of the following techniques:

(a) Remove the Highest Common Factor.

(b) If the polynomial expression consists of TWO terms (binomial expression), factorise by using one of the following rules:

- The difference of two squares.

$$a^2-b^2=(a-b)(a+b)$$

- The sum or difference of two cubes.

$$a^3+b^3=(a+b)(a^2-ab+b^2)$$
$$a^3-b^3=(a-b)(a^2+ab+b^2)$$

(c) If the polynomial expression consists of THREE terms (trinomial expression), factorise by using one of the following rules:

- Rules for perfect squares.

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- The **FOIL** method (to write expressions as linear factors).
- Completing the Square.
- Write the equation as a quadratic expression by using substitution (Let A = method).
- The factor theorem and long division.

Note: A quadratic trinomial is an expression of the form: ax^2+bx+c.

(d) If the polynomial expression consists of FOUR items, factorise using:

- The factor theorem and long division.
- Grouping two and two.
- Grouping three and one.

NUMBER SYSTEMS

Algebraic expressions are usually factorised over the following number systems:

- The rational number system (Q)
- The real number system (R)
- The integer number system (Z)

THE RATIONAL NUMBER SYSTEM (Q)

When factorising over Q, the factors can only consist of rational numbers i.e. whole numbers or numbers that may be written as fractions.

For example: $\frac{x}{2}, \frac{x+2}{3}, \frac{x+5}{x-6}, \frac{x^2+1}{2x^3-5}$ are rational numbers. Whole numbers may also be classified as rational numbers, as they too can be expressed as a fraction. Eg. $2 = \frac{2}{1}$

Note: Irrational numbers are expressions that cannot be expressed as a fraction.
For example: e, π and numbers that display an infinite number of decimal places.

THE REAL NUMBER SYSTEM (R)

When factorising over R, the factors can consist of both rational and irrational numbers (surds).

For example:

$\frac{1}{2}$, 2, $\sqrt{10}$, $\frac{1}{\sqrt{3}}$ are real.

$\sqrt{-5}$ is not part of the real number system as the square root of a negative number cannot be evaluated.

THE INTEGER NUMBER SYSTEM (Z)

The integer number system includes positive and negative whole numbers, as well as zero.
i.e. $\ldots -3, -2, -1, 0, 1, 2, 3\ldots$

HIGHEST COMMON FACTORS

The highest common factor (HCF) is the highest single term that can be taken outside a bracket. The HCF includes the largest common numeric factor (number), and the highest common power of each pronumeral (letter).

Common factors should be removed before applying any other technique in factorisation. Not only does the removal of a common factor make the factorisation process simpler, it may also help to reveal what further techniques in factorisation need to be applied.

METHOD:

Step 1: Identify the numeric factors and letters that are common to all terms.

Step 2: Write the common factor outside a set of brackets.

Step 3: Write the given equation inside the brackets and divide each term by the highest common factor.

Step 4: Simplify.

QUESTION 9

Factorise $4a^2 + 6ab$.

Solution

Identify the HCF:

The highest term that divides into both $4a^2$ and $6ab$ is $2a$.

Write the HCF outside a set of brackets:

$2a(\quad)$

Divide each term by the HCF and simplify:

$$4a^2 + 6ab = 2a\left(\frac{4a^2}{2a} + \frac{6ab}{2a}\right) = 2a(2a + 3b)$$

QUESTION 10

Factorise $10xy - 5xy^2 + 5x^2y$.

Solution

The HCF is $5xy$.

$$\therefore 10xy - 5xy^2 + 5x^2y = 5xy\left(\frac{10xy}{5xy} - \frac{5xy^2}{5xy} + \frac{5x^2y}{5xy}\right)$$

$$= 5xy(2 - y + x)$$

QUESTION 11
Factorise the following expressions.

(a) $8p^5q^2+16p^6q^3-12p^4q^7$

(b) $x+\dfrac{1}{x}$

(c) $(x-5)^2-2x+10$

(d) $25a^2(a-6)-10a(a-6)^2$

THE DIFFERENCE OF TWO SQUARES (DOTS)

$$a^2 - b^2 = \left(\sqrt{a^2} + \sqrt{b^2}\right)\left(\sqrt{a^2} - \sqrt{b^2}\right) = (a+b)(a-b)$$

The Difference of Two Squares (**DOTS**) is used to factorise equations that consist of the difference of two terms (binomial expressions), both of which are perfect squares.

To factorise these expressions:

Step 1: Remove the highest common factor.
Step 2: Take the square root of each entire term.
Step 3: Add and subtract each term.

Note: The sum of two squares ($a^2 + b^2$) cannot be factorised.

QUESTION 12
Factorise each of the following expressions:

(a) $a^2 - b^2$

$$\begin{aligned} a^2 - b^2 &= {}^{+}\sqrt{a^2} \pm {}^{+}\sqrt{b^2} \\ &= a \pm b \\ &= (a+b)(a-b) \end{aligned}$$

(b) $4x^2 - 16y^2$

$$\begin{aligned} 4x^2 - 16y^2 &= 4(x^2 - 4y^2) \\ &= 4(\sqrt{x^2} \pm \sqrt{4y^2}) \\ &= 4(x \pm 2y) \\ &= 4(x+2y)(x-2y) \end{aligned}$$

(c) $16 - 7y^2$

(d) $(x-1)^2-4$

(e) $(2x-1)^2-(x+1)^2$

(f) $4(x-2y)^2-9(2x+y)^2$