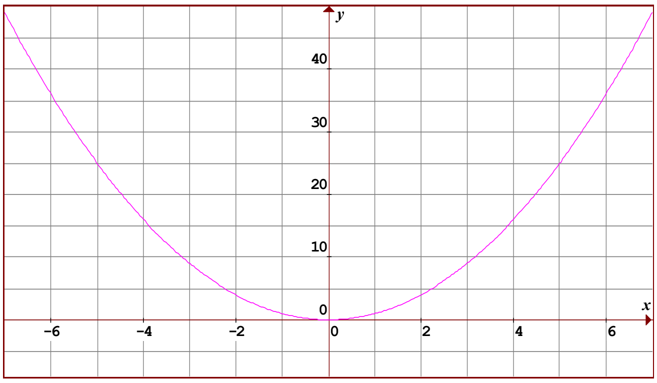
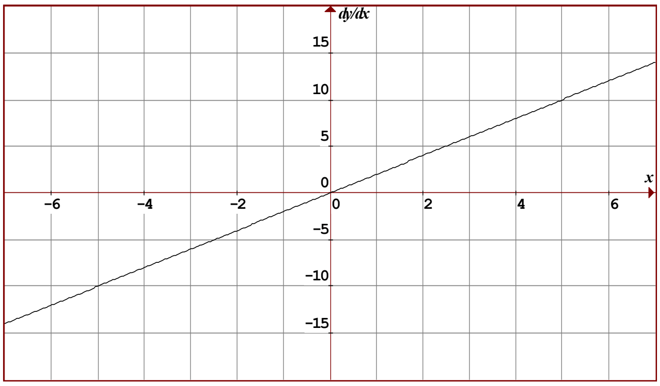
GRAPHS OF THE DERIVATIVE FUNCTION

Given the graph or equation describing y we can sketch the rate of change or gradient function $\frac{dy}{dx}$. An example of a rate graph and its original function is illustrated below.

Consider the curve $y = x^2$ and the graph of its rate of change function:





A rate of change graph provides us with a summary of the gradients or instantaneous rates of change. These graphs enable us to directly read off the gradient of the tangent at any value of x.

For example: When $x = 5$ the gradient of the tangent to y is equal to 10.

The section above the X axis in the rate graph is used to represent the regions where the graph of y has a positive gradient, i.e. $\frac{dy}{dx} > 0$. In the example above, this occurs for values of x between 0 and ∞.

Similarly, the section below the X axis in the rate graph is used to represent the regions where the graph of y has a negative gradient, i.e. $\frac{dy}{dx} < 0$. This occurs for values of x between $-\infty$ and 0.

The X intercepts on the rate graph represent the regions where the gradient of the tangent to the curve y is equal to zero, i.e. $\frac{dy}{dx} = 0$. These regions correspond to the **turning points** on the graph of y. It is at these points where the gradient of a curve changes sign.

Magnitudes

Not only is the sign of the gradient important when sketching rate graphs, the steepness or the **magnitude** of the gradients also provides valuable information of how a rate graph will appear.

Consider the regions of the graph of y which have a positive gradient. As x approaches ∞, the steepness of the tangents increase.

Similarly, as x approaches $-\infty$ (from $x = 0$), the gradient of the tangent increases in magnitude.

The Shape of the Rate Function

The shape of the derivative function can be accurately determined by analysing and plotting the magnitude (size) of the gradient at different values of x.

Helpful Hint:

If the rate function is modelled by a polynomial expression, then its degree is one less than that of the original function!

Therefore:

- Quartic functions become cubic functions.
- Cubic functions become quadratic functions.
- Quadratic functions become linear functions.

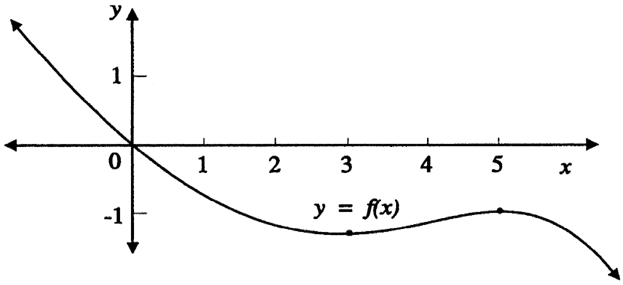
SUMMARY – SKETCHING RATE GRAPHS

To sketch accurate rate of change graphs remember to consider both the sign and the magnitude of the gradients. Mark in the turning points first and then evaluate the behaviour of the gradients on either side.

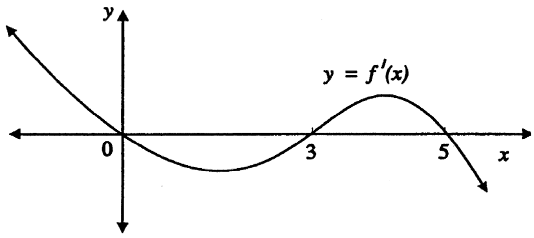
Graph of y	**Graph of $\frac{dy}{dx}$**
Positive gradient	Sketch above the X axis
Negative Gradient	Sketch below the X axis
Turning Point (Gradient $= 0$)	X intercepts

QUESTION 61

The graph of f is shown here. Which one of the following could be the graph of the derivative?



A.



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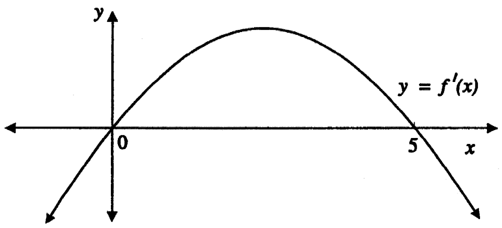
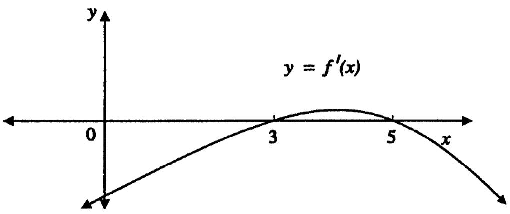
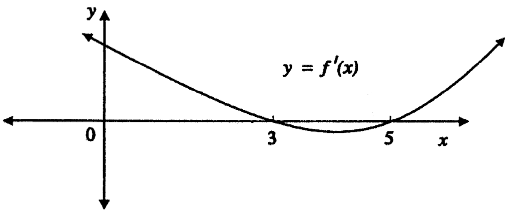


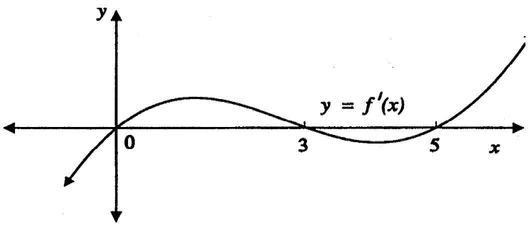
C.



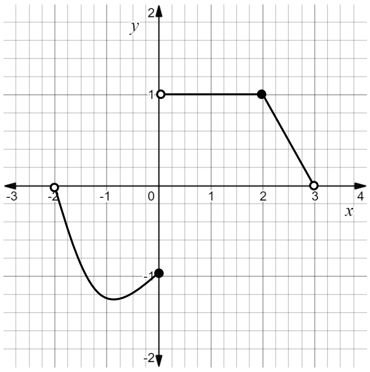
D.



E.

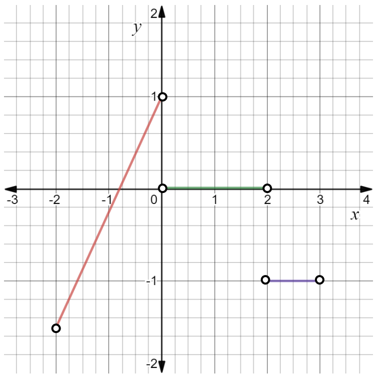


QUESTION 62
The graph of the function f is shown below.

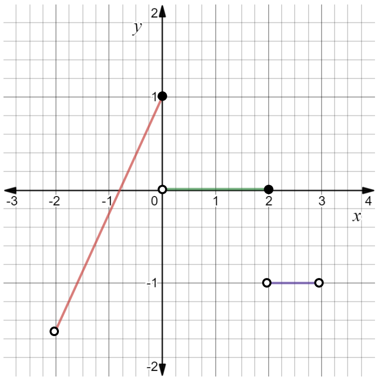


The graph of the derived function, f' is best represented by:

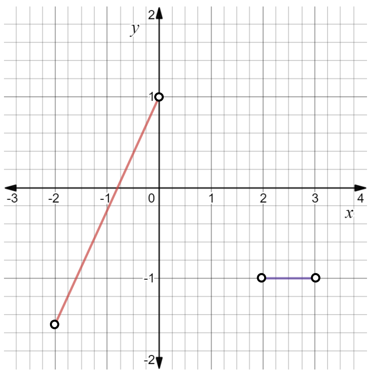
A



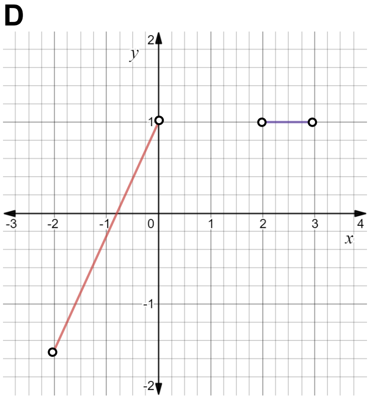
B



C



D



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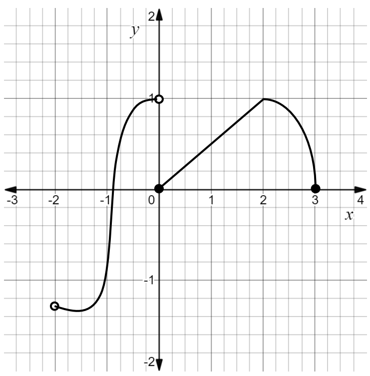
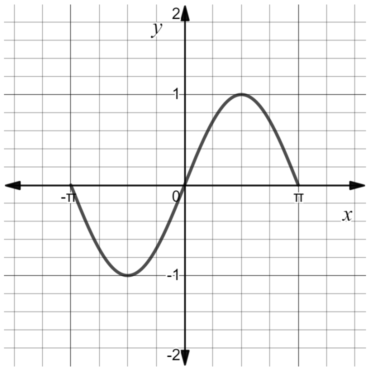
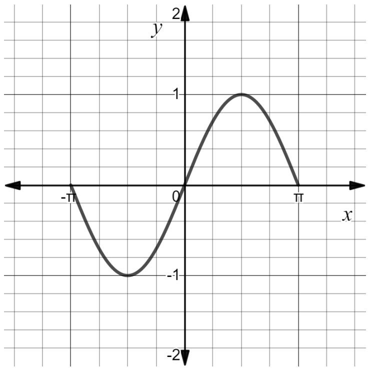


QUESTION 63
The graph of the function $f:(-\pi,\pi)\to R$ is shown below.

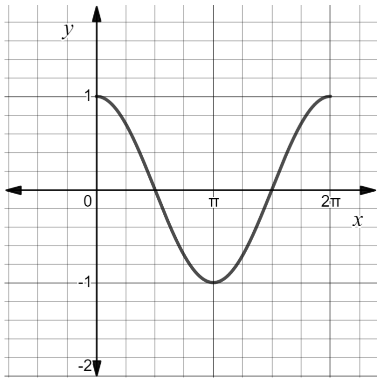


The graph of the derived function, f' is best represented by:

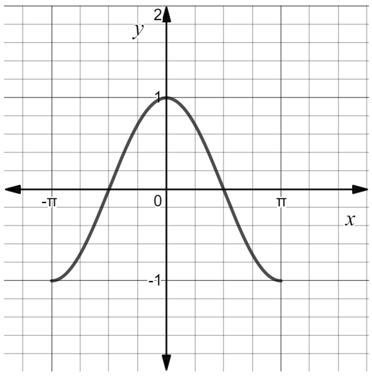
A



B



C



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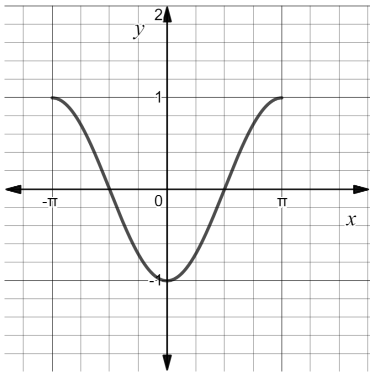

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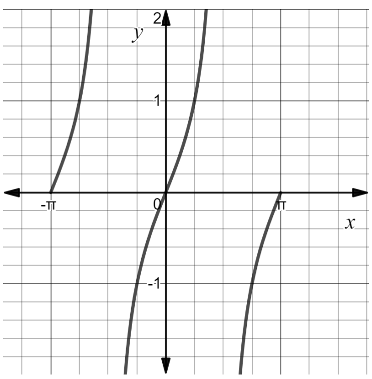


TOPIC 5: ANTIDIFFERENTIATION

Antidifferentiation is the opposite process to differentiation. We use techniques in antidifferentiation when we have been given the equation describing the derivative, and the equation for y or $f(x)$ is required.

Notations

- The symbol $\int$ indicates that the given expression is to be antidifferentiated.
- $\int(\textbf{\textit{expression}})\,dx$ indicates that the given expression is to be antidifferentiated with respect to x.

INTEGRATING ALGEBRAIC EXPRESSIONS

To antidifferentiate algebraic expressions, we raise the power on the variable (usually x) by one, and divide the new term by the new power.

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \quad n \neq -1$$

This rule applies for all algebraic expressions, including rational functions, providing that $n \neq -1$.

For Example: $\int x^3\,dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$

The antiderivative of an expression consisting of the sum/difference of a series of terms is equivalent to the sum/difference of the antiderivatives of each individual term.

For Example: $\int\left(x^3 - 2x^2 + 1\right)dx = \int\left(x^3\right)dx - \int\left(2x^2\right)dx + \int 1\,dx$

Note:

- **c** is an arbitrary constant that must be included every time an expression is antidifferentiated. The only time the arbitrary constant can be omitted is when **"an antiderivative"** is required.
- Expressions may be simplified by removing constants and placing them in front of the symbol of integration.

  For Example: $\int\left(2x^3\right)dx = 2\int\left(x^3\right)dx$.
- If asked for "an" antiderivative, no "c" is required.

METHOD:

Step 1: Rewrite all terms as powers on x.

For example: $\int\left(x^3 + 6\sqrt{x}\right)dx = \int\left(x^3 + 6x^{\frac{1}{2}}\right)dx$

Step 2: Bring terms involving x in the denominator (bottom of a fraction) to the top, by changing the sign on the power.

For example: $\frac{1}{x^2} = x^{-2}$. Note: $\frac{1}{6x^2} = \frac{x^{-2}}{6}$.

Step 3: Simplify expressions so that terms are separated by addition and subtraction and then antidifferentiate each term individually. Alternatively, reduce expressions down to 1 term only (use log and index laws).

Step 4: Antidifferentiate.

Step 5: Re-write the answer using positive powers. Bring terms with negative powers in the numerator (top of a fraction) to the bottom by changing the sign on the power.

Step 6: State restrictions on the values of x.

SIMPLIFYING EXPRESSIONS

Products:

- Expand if powers are small.
- The product of two or more terms with the same base may be simplified by adding powers, and reducing the given equation to one term (apply index laws).

  For Example: $\int\left(x^5 . x^{a+3}\right)dx = \int\left(x^{a+8}\right)dx$

Quotients

- Remove common factor(s) and simplify.
- Factorise and eliminate terms by cancellation.
- If two or more terms are written over the same denominator, write each term in the numerator over the denominator to produce individual fractions. Simplify each fraction and then antidifferentiate each term individually.

  For Example: $\int\left(\frac{x^3 + x}{x^2}\right)dx = \int\left(\frac{x^3}{x^2} + \frac{x}{x^2}\right)dx = \int\left(x + x^{-1}\right)dx$

- Simplify expressions using Index Laws. (Observe the use of brackets).

  For example: $\int\left(\frac{x^{2a-1}}{x^{1-4a}}\right)dx = \int x^{(2a-1)-(1-4a)}dx = \int x^{2a-1-1+4a}dx = \int x^{6a-2}dx = \frac{x^{6a-1}}{6a\text{-}1} + c$.

Note: You may check your answers at any stage by differentiating your answer!

QUESTION 1
Find $f(x)$ given that:

(a) $f'(x) = 2x - 7$

(b) $f'(x) = \frac{2}{3}x + \frac{3}{8}x^2$

(c) $f'(x) = (x^2 - 3)(x + 5)$

(d) $f'(x) = \frac{2}{x}(x^3 - 5x^2 - 7x)$

(e) $f'(x) = \sqrt{x}(x^2 + 9x - 3)$

(f) $f'(x) = \frac{6x^4 - 3x^3 + 2x^2}{2x^2}$

QUESTION 2

$\int\left(3\sqrt{x}-\frac{1}{\sqrt{x}}-1\right)dx$ is equal to:

A $3\sqrt{x}-\frac{1}{\sqrt{x}}-x+c$

B $2\sqrt{x^3}-2\sqrt{x}-1+c$

C $2\sqrt{x^3}-2\sqrt{x}-x+c$

D $\frac{3}{2\sqrt{x}}+\frac{1}{2\sqrt{x^3}}+c$

E $2\sqrt{x^3}-2\sqrt{x}+c$

QUESTION 3

$\int\left(x+\frac{1}{x}\right)^2 dx=$

A $\frac{x^3}{3}-\frac{1}{x}+c$

B $2x+\frac{x^3}{3}+\frac{3}{x^3}+c$

C $\left(\frac{x^2}{2}+\log_e x\right)^2+c$

D $2x-\frac{2}{x^3}+c$

E $\frac{x^3}{3}-\frac{1}{x}+2x+c$

SOLVING FOR THE CONSTANT C

The gradient function alone does not provide enough information to determine the antiderivative, and for this reason, we must include the arbitrary constant c as part of our solution.

To solve for the value c, the coordinates of one point on the graph of $f(x)$ must be known.

METHOD:

Step 1: Antidifferentiate the given expression.
Step 2: Solve for c by substituting the given values of x and y into the equation for $f(x)$.

QUESTION 4
Find the rule that defines a function, f, given that $f'(x) = x^2 - 2x + b$ for all x, and $f'(0) = 1$ and $f(0) = 2$.

Solution

Simplify the expression by substituting any known/given data:

When $x = 0$, the derivative $= 1$ i.e. $f'(0) = 1$:

$f'(x) = x^2 - 2x + b$
$1 = (0)^2 - 2(0) + b \quad \therefore b = 1$

Find an antiderivative: $f(x) = \frac{x^3}{3} - \frac{2x^2}{2} + x + c = \frac{x^3}{3} - x^2 + x + c$

Solve for c by substituting the given values of x and y into the equation describing $f(x)$:

$f(0) = 2$ i.e. When $x = 0, y = 2$

$2 = 0 - 0 + 0 + c$
$\therefore c = 2$

The equation describing $f(x)$ is $\frac{x^3}{3} - x^2 + x + 2$.

QUESTION 5

The area $A\ cm^2$ of a healing wound is decreasing at a rate given by $A'(t) = -45t^{-2}$ where t is the time in days. If $A(1) = 39$, find:

(a) $A(t)$.

(b) The area of the wound after 4 days.

Solution

QUESTION 6

The gradient at all points on a curve is given by $f'(x) = 1 - 4x$. If the curve passes through the point $(2, 5)$, find the coordinates of the point where the curve crosses the Y axis.

Solution

QUESTION 7

Find the equation of a curve given that $\frac{dy}{dx} = 2x + b$ at any point P, and that when $x = 3$, $\frac{dy}{dx} = 2$ and $y = -3$.

Solution

QUESTION 8

Let $f'(x) = g'(x) + 3$, $f(0) = 2$ and $g(0) = 1$. Then $f(x)$ is given by:

A $f(x) = g(x) + 3x + 1$

B $f(x) = g'(x) + 3$

C $f(x) = g(x) + 3x$

D $f(x) = 1$

E $f(x) = g(x) + 3$

Solution

DEFINITE INTEGRALS

Indefinite Integrals

Indefinite integrals are generally obtained by integrating the given equation with respect to x. The indefinite integral gives an equation that represents the general antiderivative.

Definite Integrals

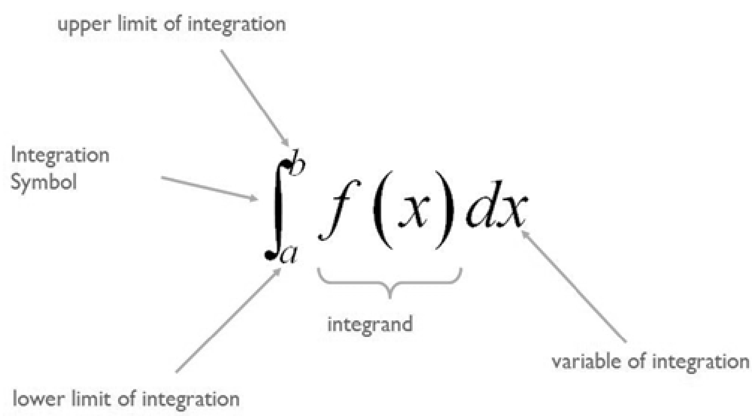
Definite integrals involve calculating the antiderivative between two particular values of x: **a** and **b.** A numeric value is generally obtained.

Definite Integrals are represented in the following format:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

Where:

- a is known as the lower limit of integration and usually represents the most negative value of x.
- b is known as the upper limit of integration and usually represents the most positive value of x.
- $F(x)$ is the antiderivative of $f(x)$.



METHOD:

Step 1: Antidifferentiate the given expression to find $F(x)$.

Step 2: Write the answer in the form $[F(x)]_a^b$.

Step 3: Substitute the upper limit (b) into the antiderivative to find $F(b)$.

Step 4: Substitute the lower limit (a) into the antiderivative to find $F(a)$.

Step 5: Subtract $F(a)$ from $F(b)$.

QUESTION 9

Using calculus, find $\int_0^2 (2x^2 - 3x - 1)\, dx$.

Solution

Step 1: Antidifferentiate the given equation to find $F(x)$.

$$\int (2x^2 - 3x - 1)\, dx = \frac{2x^3}{3} - \frac{3x^2}{2} - x + c$$

Step 2: Write the answer in the form $[F(x)]_a^b$.

$$\int (2x^2 - 3x - 1)\, dx = \left[\frac{2x^3}{3} - \frac{3x^2}{2} - x\right]_0^2$$

Step 3: Substitute the upper and lower limits and subtract $F(a)$ from $F(b)$.

$$\int_0^2 (2x^2 - 3x - 1)\, dx = \left(\frac{16}{3} - \frac{12}{2} - 2\right) - (0) = \frac{16}{3} - 8 = -\frac{8}{3}$$