CASE 4:

m and *n* are both odd and positive.

There is a choice between **CASE 2** and **CASE 3**. Make the choice that leads to the least amount of work.

CASE 5:

m and *n* are both even and positive.

Use the following identities (derived from the double angle formulae):

$$\sin(A)\cos(A) = \tfrac{1}{2}\sin(2A) \qquad \sin^2(A) + \cos^2(A) = 1$$

$$\sin^2(A) = \tfrac{1}{2}(1 - \cos(2A))$$

$$\cos^2(A) = \tfrac{1}{2}(1 + \cos(2A))$$

This will convert the given problem into separate **CASE 0** – **CASE 3** problems.

PARTIAL FRACTION DECOMPOSITION

If we add $\frac{3}{x+7}$ to $\frac{5}{x-3}$ we get:

$$\frac{3}{x+7} + \frac{5}{x-3} = \frac{3(x-3)+5(x+7)}{(x+7)(x-3)} = \frac{3x-9+5x+35}{(x+7)(x-3)} = \frac{8x+26}{(x+7)(x-3)}$$

However, there are times when we need to split a fraction such as $\frac{8x+26}{(x+7)(x-3)}$ into its separate fractional parts, that is, into $\frac{3}{x+7} + \frac{5}{x-3}$.

This process is called partial fraction decomposition. It is done in order to anti-differentiate rational functions.

CAS CAN DO THIS FOR YOU

$$\text{expand}\left(\frac{8\cdot x+26}{(x+7)\cdot(x-3)}, x\right)$$

$$\frac{3}{x+7} + \frac{5}{x-3}$$

RATIONAL FUNCTIONS

A rational function is any function of the form $\frac{N(x)}{D(x)}$, where $N(x)$ and $D(x)$ are polynomials.

In the VCE Specialist Mathematics 3-4 course, only rational functions where $D(x)$ = quadratic are considered.

N(x) = constant or N(x) = linear (that is, the degree of N(x) is less than the degree of D(x)).

Step 1: Factorise $D(x)$ into its real linear factors.

Step 2: Express $\frac{N(x)}{D(x)}$ as the sum of two fractions. This is the partial fraction decomposition of $\frac{N(x)}{D(x)}$.

Type 1:

$D(x)$ has two distinct linear factors, that is $D(x) = (ax+b)(cx+d)$.

Then $\frac{N(x)}{D(x)} = \frac{A}{ax+b} + \frac{B}{cx+d}$, where *A* and *B* are constants.

Type 2:

$D(x)$ has repeated linear factors, that is, $D(x) = (ax+b)^2$.

Then $\frac{N(x)}{D(x)} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$, where *A* and *B* are constants.

Type 3:

$D(x)$ has no real linear factors, that is, $D(x)$ is an irreducible quadratic.

Then $\frac{N(x)}{D(x)}$ cannot be expressed in partial fraction form, unless there is another factor multiplied by it.

Step 3: Determine the values of the constants *A* and *B* by comparing $\frac{N(x)}{D(x)}$ with its partial fraction form (re-write over a common denominator and equate numerators).

PARTIAL FRACTIONS

TYPE I: FACTORISABLE DENOMINATOR

$$\frac{something}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$$

Example:

$$\frac{2}{x^2+4x+3} = \frac{2}{(x+1)(x+3)} = \frac{A}{(x+1)} + \frac{B}{(x+3)}$$

Giving:

$$\frac{2}{(x+1)(x+3)} = \frac{A(x+3)+B(x+1)}{(x+1)(x+3)}$$

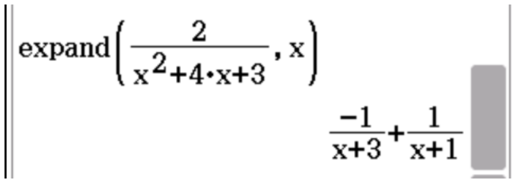
Equate numerators:

$$1 = A(x+3)+B(x+1)$$

Let $\begin{cases} x=-3 \therefore 2=-2B \therefore B=-1 \\ x=-1 \therefore 2=2A \therefore A=1 \end{cases}$

Giving $\dfrac{2}{x^2+4x+3} = \dfrac{1}{(x+1)} - \dfrac{1}{(x+3)}$

On a CASIO ClassPad:



TYPE II: REPEATED FACTOR ON DENOMINATOR

$$\frac{something}{(x+a)^2(x+b)}=\frac{A}{(x+a)^2}+\frac{B}{(x+a)}+\frac{C}{(x+b)}$$

Example:

$$\frac{2x}{(x+1)^2(x+3)}=\frac{A}{(x+1)^2}+\frac{B}{(x+1)}+\frac{C}{(x+3)}$$

Giving:

$$\frac{2x}{(x+1)^2(x+3)}=\frac{A(x+3)+B(x+1)(x+3)+C(x+1)^2}{(x+1)^2(x+3)}$$

Equate numerators:

$$2x=A(x+3)+B(x+1)(x+3)+C(x+1)^2$$

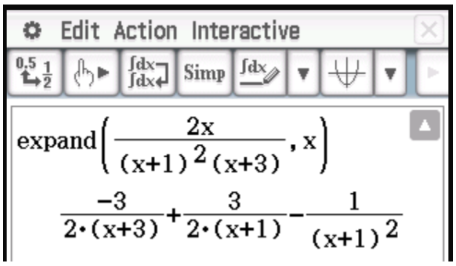
Let $\begin{cases} x=-3 \therefore -6=4C \therefore C=-\frac{3}{2} \\ x=-1 \therefore -2=2A \therefore A=-1 \\ x=0 \therefore 0=3A+3B+C \end{cases}$

Substitute and solve $0=3A+3B+C$ to get $0=-3+3B-\frac{3}{2}$

Giving $A=-1, B=\frac{3}{2}, C=-\frac{3}{2}$

Giving $\frac{2x}{(x+1)^2(x+3)}=\frac{-1}{(x+1)^2}+\frac{3}{2(x+1)}-\frac{3}{2(x+3)}$

On a CASIO ClassPad:



TYPE III: NON-REDUCIBLE FACTOR ON DENOMINATOR

$$\frac{something}{\left(ax^2+bx+c\right)\left(x+d\right)}=\frac{Ax+B}{ax^2+bx+c}+\frac{C}{x+d}$$ in the case where $\Delta\left(ax^2+bx+c\right)<0$

Example 1

$$\frac{2x+1}{\left(x^2+x+2\right)\left(x+5\right)}=\frac{Ax+B}{\left(x^2+x+2\right)}+\frac{C}{\left(x+5\right)}$$ where $\Delta\left(x^2+x+2\right)=1-4(1)(2)=-7<0$

Giving:

$$\frac{2x+1}{\left(x^2+x+2\right)\left(x+5\right)}=\frac{\left(Ax+B\right)\left(x+5\right)+C\left(x^2+x+2\right)}{\left(x^2+x+2\right)\left(x+5\right)}$$

Equate numerators:

$$2x+1=\left(Ax+B\right)\left(x+5\right)+C\left(x^2+x+2\right)$$

Let $\begin{cases} x=-5 \therefore -9=22C \therefore C=-\frac{9}{22} \\ x=1 \therefore 3=6\left(A+B\right)+4C \\ x=0 \therefore 1=5B+2C \end{cases}$

Substitute and solve $1=5B+2C$ to get $1=5B-\frac{9}{11} \therefore B=\frac{4}{11}$

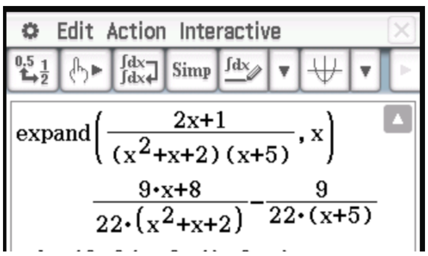
Substitute and solve $3=6\left(A+B\right)+4C$ to get $3=6\left(A+\frac{4}{11}\right)-\frac{18}{11} \therefore A=\frac{9}{22}$

Giving $A=\frac{9}{22}, B=\frac{4}{11}, C=-\frac{9}{22}$

Giving $$\frac{2x+1}{\left(x^2+x+2\right)\left(x+5\right)}=\frac{\frac{9}{22}x+\frac{4}{11}}{\left(x^2+x+2\right)}-\frac{\frac{9}{22}}{\left(x+5\right)}$$

OR $$\frac{2x+1}{\left(x^2+x+2\right)\left(x+5\right)}=\frac{9x+8}{22\left(x^2+x+2\right)}-\frac{9}{22\left(x+5\right)}$$

On a CASIO ClassPad:



Example 2

$$\frac{x}{\left(x^2+2x+3\right)\left(x+1\right)}=\frac{Ax+B}{\left(x^2+2x+3\right)}+\frac{C}{\left(x+1\right)}$$ where $\Delta\left(x^2+2x+3\right)=4-4(1)(3)=-8<0$

Giving

$$\frac{x}{\left(x^2+2x+3\right)\left(x+1\right)}=\frac{\left(Ax+B\right)\left(x+1\right)+C\left(x^2+2x+3\right)}{\left(x^2+2x+3\right)\left(x+1\right)}$$

Equate numerators

$$x=\left(Ax+B\right)\left(x+1\right)+C\left(x^2+2x+3\right)$$

Let $\begin{cases} x=-1 \therefore -1=2C \therefore C=-\dfrac{1}{2} \\ x=1 \therefore 1=2\left(A+B\right)+6C \\ x=0 \therefore 0=B+3C \end{cases}$

Substitute and solve $0=B+3C$ to get $0=B-\frac{3}{2} \therefore B=\frac{3}{2}$

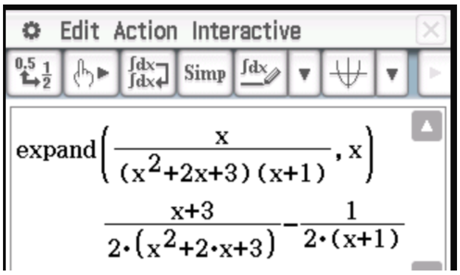
Substitute and solve $1=2\left(A+B\right)+6C$ to get $1=2\left(A+\frac{3}{2}\right)-3 \therefore A=\frac{1}{2}$

Giving $A=\frac{1}{2}, B=\frac{3}{2}, C=-\frac{1}{2}$

Giving $\dfrac{x}{\left(x^2+2x+3\right)\left(x+1\right)}=\dfrac{\frac{1}{2}x+\frac{3}{2}}{\left(x^2+x+2\right)}-\dfrac{\frac{1}{2}}{\left(x+5\right)}$

OR $\dfrac{x}{\left(x^2+2x+3\right)\left(x+1\right)}=\dfrac{x+3}{2\left(x^2+x+2\right)}-\dfrac{1}{2\left(x+5\right)}$

On a CASIO ClassPad:



N(X) = POLYNOMIAL OF DEGREE 2 OR HIGHER

(For example, quadratic, cubic etc.) (That is, the degree of N(x) is greater than or equal to the degree of D(x).)

Step 1: Use polynomial long division:

$$D(x)\overline{\big)N(x)}\quad\text{quotient } Q(x),\ \vdots,\ \text{remainder } R(x) \qquad \Rightarrow \frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)},$$

Where $R(x)$ will be either a constant or a linear.

Step 2: Express $\frac{R(x)}{D(x)}$ in partial fraction form.

SUMMARY

Expression	Partial Fractions	Examples
$\frac{ax+b}{(cx+d)(ex+f)}$	$\frac{A}{cx+d}+\frac{B}{ex+f}$	$\frac{3x-1}{(x-2)(2x-3)} \equiv \frac{A}{(x-2)}+\frac{B}{(2x-3)}$
$\frac{ax+b}{(px+q)^2}$	$\frac{A}{(px+q)}+\frac{B}{(px+q)^2}$	$\frac{5x+2}{(x-3)^2} \equiv \frac{A}{(x-3)}+\frac{B}{(x-3)^2}$
$\frac{ax+b}{(mx^2+nx+r)(px+q)}$ Where mx^2+nx+r is a non reducible quadratic	$\frac{Ax+B}{(mx^2+nx+r)}+\frac{C}{(px+q)}$	$\frac{2x+1}{(3x^2+x+2)(3x+1)} = \frac{Ax+B}{(3x^2+x+2)}+\frac{C}{(3x+1)}$

If the denominator of a rational expression is not factorised, then factorise it first before splitting it into partial fractions.

E.g. $\frac{1-2x}{x^2-3x-10} \equiv \frac{A}{(x-5)}+\frac{B}{(x+2)}$

If the degree of the numerator of a rational expression is greater than or equal to the denominator, then divide the denominator into the numerator first before splitting the fractional part into partial fractions.

E.g. $\frac{3x^3+7x^2-4x+5}{x^2+3x-4} \equiv 3x-2+\frac{14x-3}{x^2+3x-4} \equiv 3x-2+\frac{A}{(x+4)}+\frac{B}{(x-1)}$