INDEX – UNIT 3 TSFX REFERENCE MATERIALS 2020 – 2021

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SOLVING TRIGONOMETRIC EQUATIONS

Step 1: Write all expressions in terms of one trigonometric function.

Step 2: Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation.

Step 3: Use the sign in front of the constant on the right-hand side to determine the quadrants in which the solutions are to lie. (Use **CAST**)

Step 4: Calculate the first quadrant solution. If the exact value cannot be determined:

Press Inverse Sin, Cos or Tan of the number on the right-hand side of the equation (but ignore the sign).

For example: $Sin^{-1}(number\ on\ RHS\ of\ equation\ but\ ignore\ the\ sign)$

(Ensure that the calculator is in Radian Mode).

Step 5: Solve for the variable (usually x or θ). Let the angle equal the rule describing angles in the quadrants in which the solutions are to lie.

Note: First Quadrant Angle = FQA

Let angle = FQA if solution lies in 1st Quadrant.
Let angle = $\pi - FQA$ if solution lies in 2nd Quadrant.
Let angle = $\pi + FQA$ if solution lies in 3rd Quadrant.
Let angle = $2\pi - FQA$ if solution lies in 4th Quadrant.

Step 6: Evaluate all possible solutions by observing the given domain. This is accomplished by adding or subtracting the **PERIOD** to each of the solutions, until the angles fall outside the given domain.

For sine and cosine functions: $Period = \frac{2\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

For tangent functions: $Period = \frac{\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

Always look closely at the brackets in the given domain and consider whether the upper and lower limits can be included in your solutions.

DO NOT discard any solution until the final step.

Step 7: Eliminate solutions that do not lie across specified domain.

Note: Students may also solve trigonometric equations by rearranging the domain.

Example:

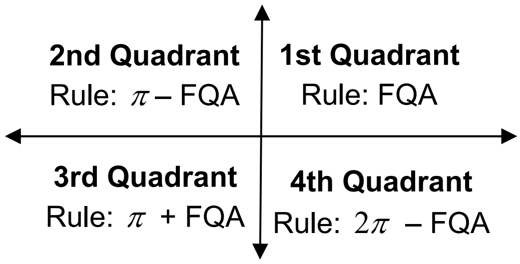
Solve $\frac{2}{\sqrt{3}}\sin 2\left(x-\frac{\pi}{3}\right)=-1$ for $x\in[-\pi,\pi]$.

Solution

$$\frac{2}{\sqrt{3}}\sin 2\left(x-\frac{\pi}{3}\right)=-1$$

$$\sin 2\left(x-\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

$$\text{Sin}^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$$



Solutions are required in the quadrants where sine is negative i.e. Quadrants 3 and 4.

Quadrant 3 Rule: $\pi + First\ Quadrant\ Angle = \pi+\frac{\pi}{3}=\frac{4\pi}{3}$

$$\therefore 2\left(x-\frac{\pi}{3}\right)=\frac{4\pi}{3}$$

$$\therefore \left(x-\frac{\pi}{3}\right)=\frac{2\pi}{3}$$

$$x=\frac{2\pi}{3}+\frac{\pi}{3}$$

$$x=\pi$$

Quadrant 4 Rule: $2\pi - First\ Quadrant\ Angle = 2\pi-\frac{\pi}{3}=\frac{5\pi}{3}$

$$\therefore 2\left(x-\frac{\pi}{3}\right)=\frac{5\pi}{3}$$

$$\therefore \left(x-\frac{\pi}{3}\right)=\frac{5\pi}{6}$$

$$x=\frac{5\pi}{6}+\frac{2\pi}{6}$$

$$x=\frac{7\pi}{6}$$

Add and subtract the period to each solution: $T=\pi$

$$x=-\pi,-\frac{5\pi}{6},0,\frac{\pi}{6},\pi$$

WATCHOUTS

- Remember to add/subtract the period to each one of your solutions – making sure that you do not exceed domain. To avoid mistakes – write each term to the same denominator.
- Never ever eliminate an answer until the final step (so as to assure that all solutions are obtained).
- Look closely at brackets around the domain and assess whether the first and last solution can be included.
- Given a physical/real life situation, pay close attention to the domain. Consider the real life limitations on your solutions. For example - lengths cannot be negative.
- If number on right hand side ends up being 0 or $\pm$ 1, use the unit circle to find number of solutions per period. For all other numbers – you will get two solutions per period.
- You cannot find the inverse sine or cosine of a number greater than 1 or less than –1.

  i.e. $\sin\theta =$ number that lies between –1 and +1 inclusive.
  $\cos\theta =$ number that lies between –1 and +1 inclusive.
  $\tan\theta =$ any real number

- You cannot solve two trigonometric expressions that have different angles algebraically (use technology) unless you can find one expression in terms of the other using complementary rules.
- You cannot convert an equation containing a sin and cos to tan unless they share the same angle.
- What does $\sin 2\left(x+\frac{\pi}{3}\right)=0.5$ find???

  Answer: The points of intersection of $y=\sin 2\left(x+\frac{\pi}{3}\right)$ and $y=0.5$.

- What does $\sin 2\left(x+\frac{\pi}{3}\right)-0.5=0$ find???

  Answer: The X intercepts on the graph of $y=\sin 2\left(x+\frac{\pi}{3}\right)-0.5$.

- Given an inequation – solve the equation without the inequality and then reason from the graph.
- When solving questions to a given number of decimal places – make sure the calculator is in RADIAN mode.

SOLVING COMPLEX TRIGONOMETRIC EQUATIONS

Students are required to be able to manipulate expressions in terms of two or more different trigonometric functions, as well as solve questions involving both trigonometric functions and other expressions such as logarithmic, exponential and polynomial functions.

To solve expressions written in terms of two or more trigonometric functions, apply one of the following techniques.

If the angles are the same:

- Simplify equations by removing common factors.

  eg. $\cos^2\theta - \sin\theta\cos\theta = \cos\theta\left(\cos\theta - \sin\theta\right)$

- If the expression is presented in its factorised form (or can be factorised) and is equal to zero, apply the **null factor law** to obtain solutions.

  eg. $\cos\theta\left(\cos\theta - \sin\theta\right) = 0$

  $\therefore \cos\theta = 0 \ \ and \ \ \cos\theta - \sin\theta = 0$

- Given both a sine and cosine function – write each function on either side of the equality sign. Convert the expression to a tangent function by dividing both sides by cos or sin.

  eg. $\sin 4x + \cos 4x = 0$

  $\therefore \sin 4x = -\cos 4x$

  $\therefore \dfrac{\sin 4x}{\cos 4x} = -\dfrac{\cos 4x}{\cos 4x}$

  $\therefore \tan 4x = -1$

- Given two or more terms involving the same trigonometric function (but each with different powers) apply quadratic factors ("Let A =" method).

  eg. $3\sin^6(5x) - 2\sin^3(5x) - 1 \quad \therefore Let\ A = \sin^3(5x)$

  $\therefore 3\left(\sin^3(5x)\right)^2 - 2\sin^3(5x) - 1 \quad \therefore 3A^2 - 2A - 1$

These techniques can only be successfully applied at this level of mathematics if the angles of each of the trigonometric expressions are identical.

If the angles are different:

- Use complementary or supplementary rules to write one angle in terms of the other or to write mixtures of trigonometric functions with different angles to the same trigonometric expression with different angles. These expressions can then be solved by **EQUATING** angles. eg. $\sin(3x) = \cos\left(x + \frac{\pi}{4}\right)$

  $\therefore \cos\left(\frac{\pi}{2} - 3x\right) = \cos\left(x + \frac{\pi}{4}\right)$

  $\left(\frac{\pi}{2} - 3x\right) = \left(x + \frac{\pi}{4}\right)$

  $\therefore 4x = \frac{\pi}{4}$

  $\therefore x = \frac{\pi}{16} + nT$

- Otherwise, use the **SOLVE** or **INTERSECT** function on your calculator to find solutions.

To solve questions involving both trigonometric functions and other expressions such as logarithmic, exponential and polynomial functions:

Use the **SOLVE** or **INTERSECT** function on your calculator to find solutions.

For example: Solve $2\sin\left(x - \frac{\pi}{2}\right) = 3e^{6x-1}$.

SOLVING TRIGONOMETRIC EQUATIONS

QUESTION 12

Solve $2\sin 2\left(x+\frac{\pi}{3}\right)=\sqrt{3}$, $x\in[0, 2\pi]$ across the given domain.

Solution

Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation:

$$\sin 2\left(x+\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$$

Calculate the first quadrant solution:

1st Quadrant Angle $= Sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$

Use the sign in front of the constant on the right hand side to determine the quadrants in which the solutions are to lie:

Solutions are to lie in the quadrants where sine is positive i.e. the 1st and 2nd quadrants:

S✓	A✓
T	C

Solve for the variable (usually x). Let the actual angle in the given equation equal the quadrant rules in which the solutions are to lie.

Let $2\left(x+\frac{\pi}{3}\right)=\frac{\pi}{3}$

Let $2\left(x+\frac{\pi}{3}\right)=\pi-\frac{\pi}{3}$

$$2\left(x+\frac{\pi}{3}\right)=\frac{\pi}{3}, \frac{2\pi}{3}$$

$$\left(x+\frac{\pi}{3}\right)=\frac{\pi}{6}, \frac{2\pi}{6}$$

$$x=-\frac{\pi}{6}, 0$$

Evaluate all possible solutions by observing the given domain. Add and subtract the PERIOD to each of the solutions, until the angles fall outside the given domain:

$$T=\frac{2\pi}{2}=\pi=\frac{6\pi}{6}$$

$$\left\{x : x=0, \frac{5\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi\right\}$$

QUESTION 13

Which of the following gives the possible solution(s) to the equation $\sin(2x) = \frac{1}{2}\sin\left(\frac{\pi}{2}\right)$?

I $\frac{\pi}{6}$ II $\frac{\pi}{3}$ III $\frac{\pi}{8}$ IV $\frac{\pi}{12}$

A I only
B II only
C III only
D I and II
E IV only

Solution

QUESTION 14

Find exact solutions of $\sqrt{2}\cos\left(x-\frac{3\pi}{4}\right)+1=0$ for $x\in[2\pi, 6\pi]$.

Solution

QUESTION 15

Four solutions to $2\cos\left(\frac{x}{a}\right)+\sqrt{3}=0$, where $1<a<3$, are

A $x=-\frac{11\pi a}{6}$ $x=-\frac{\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{23\pi a}{6}$

B $x=-\frac{7\pi a}{6}$, $x=-\frac{5\pi a}{6}$ $x=\frac{17\pi a}{6}$ and $x=\frac{19\pi a}{6}$

C $x=-\frac{13\pi a}{6}$ $x=-\frac{11\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{23\pi a}{6}$

D $x=-\frac{19\pi a}{6}$ $x=-\frac{13\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{11\pi a}{6}$

E $x=-\frac{17\pi a}{6}$ $x=-\frac{\pi a}{6}$, $x=\frac{\pi a}{6}$ and $x=\frac{17\pi a}{6}$

Solution

QUESTION 16

A solution of the equation $\sin(3x) = k\cos(3x)$ is $\frac{3\pi}{4}$. The value of k is:

A -2

B -1

C 0

D 1

E 2

Solution

QUESTION 17

The equation $a\cos(x+b) = c,$ where a, b, c are positive constants, will not have any solutions in the interval $[0, 2\pi)$ provided that:

A $c = a$

B $b < \frac{\pi}{2}$

C $c > 1$

D $a < c$

E $b > a$

Solution

QUESTION 18

Let $f(x) = a\sin(2x)$ and $g(x) = b$, where $0 < x < \frac{3\pi}{2}$ and a and b are positive integers.

Which of the following statements is **not** true?

A If $b > a$ there are no real solutions to the equation $f(x) = g(x)$.

B If $0 < b < a$ there are 4 solutions to the equation $f(x) = g(x)$.

C If $0 < b < a$ there are 4 solutions to the equation $f(x) = -g(x)$.

D If $a = b$ there is 1 solution to the equation $f(x) = -g(x)$.

E If $a = b$ there are 2 solutions to the equation $f(x) = g(x)$.

Solution