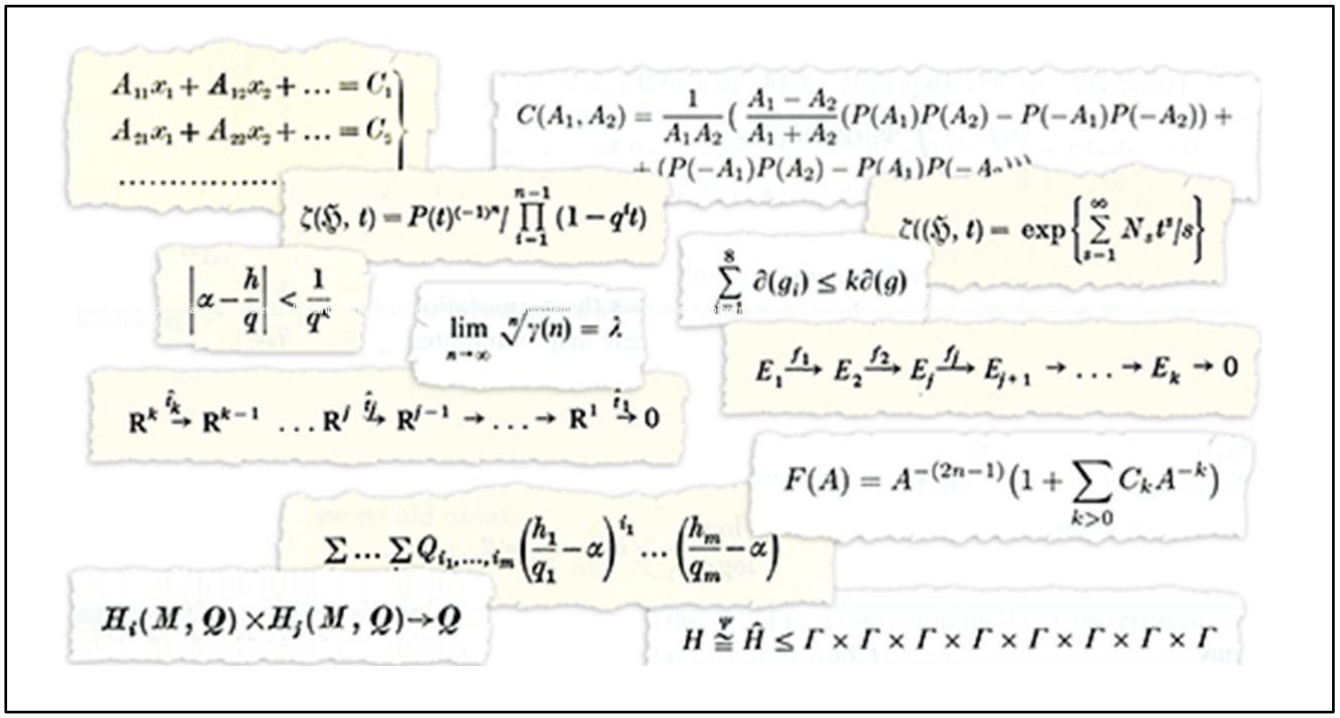
UNIT 3 SPECIALIST MATHS

DIFFERENTIAL EQUATIONS



WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

VIC: Level 3, 99 William Street, Melbourne VIC 3000
Phone: (03) 9663 3311 Fax: (03) 9663 3939

www.tsfx.com.au
admin@tsfx.com.au

NSW: PO Box r1407, Royal Exchange, Sydney NSW 1225
Phone: 1300 364 173 Fax: 1300 364 065

Chapter 9 Differential Equations

Exercise 9A An introduction to differential equations

· e.g. 1 Let $y = \arctan(2x)$. Find the value of a given that $\frac{d^2y}{dx^2} = ax\left(\frac{dy}{dx}\right)^2$, where a is a real constant. (VCAA 2012 Exam 1 Q5)

$\rightarrow \frac{dy}{dx} = \frac{1}{1+(2x)^2} \times 2 = \frac{2}{1+4x^2}$

$\frac{d^2y}{dx^2} = -2(1+4x^2)^{-1} \times 8x = \frac{-16x}{(1+4x^2)^2}$

$\rightarrow \frac{-16x}{(1+4x^2)^2} = ax\left(\frac{2}{1+4x^2}\right)^2 = \frac{4ax}{(1+4x^2)} \rightarrow \boxed{a = -4}$

· e.g. 2. Assume that $\frac{dx}{dy}$ is inversely proportional to y. Given that when $x = 0$, $y = 2$ and when $x = 2$, $y = 4$, find y when $x = 3$.

$\rightarrow \frac{dx}{dy} \propto \frac{1}{y} \rightarrow \frac{dx}{dy} = \frac{k}{y} \rightarrow x = k\log_e(y) + c$

When $x = 0$, $y = 2$ $\rightarrow 0 = k\log_e(2) + c$ ... ①

When $x = 2$, $y = 4$ $\rightarrow 2 = k\log_e(4) + c$

$2 = 2k\log_e(2) + c$ ... ②

② − ①: $2 = k\log_e(2) \rightarrow k = \frac{2}{\log_e(2)}$

Sub into ① $0 = 2 + c \rightarrow c = -2$

$\rightarrow x = \frac{2\ln(y)}{\ln(2)} - 2$

Sub $x = 3$, $3 = \frac{2\ln(y)}{\ln(2)} - 2 \rightarrow 2\ln(y) = 5\ln(2)$

$\ln(y) = \frac{5}{2}\ln(2) \rightarrow y = 2^{5/2} = \boxed{4\sqrt{2}}$

Exercise 9B Differential equations involving a function of independent variable

→ Solution of DE's (Case 1 + 2)

· $\frac{dy}{dx} = f(x) \rightarrow y = \int f(x)\,dx = F(x) + c$ (where $F'(x) = f(x)$)

· $\frac{d^2y}{dx^2} = f(x) \rightarrow \frac{dy}{dx} = \int f(x)\,dx = F(x) + c$ (where $F'(x) = f(x)$)

$\rightarrow y = \int (F(x) + c)\,dx = G(x) + cx + d$ (where $G'(x) = F(x)$)

→ CAS skills (Solving differential equations):

→ Menu → Calculus (4) → Differential Equation Solver (D)

① deSolve($y' = (x)$ or $y'' = (x)$, x, y) ($\boxed{'}$ from $\boxed{\pi >}$)

② deSolve($y'' =$ ... and $y(a) = b$ and $y'(c) = d$, x, y)

IV DV

→ Differential equations examples

e.g. 1. Consider the differential equation $\frac{d^2y}{dx^2} = \frac{4x}{(1-x^2)^2}$, $-1 < x < 1$, for which $\frac{dy}{dx} = 3$ when $x = 0$, and $y = 4$ when $x = 0$. Given that $\frac{d}{dx}\left(\frac{2}{1-x^2}\right) = \frac{4x}{(1-x^2)^2}$, find the solution of this differential equation. (VCAA 2010 Exam 1 Q7)

$\rightarrow \frac{dy}{dx} = \int \frac{4x}{(1-x^2)^2}\,dx = \frac{2}{1-x^2} + c$

When $x = 0$, $\frac{dy}{dx} = 3 \rightarrow 3 = \frac{2}{1-0} + c \rightarrow c = 1$

$\rightarrow y = \int \left(\frac{2}{1-x^2} + 1\right) dx = \int \left(\frac{1}{1+x} + \frac{1}{1-x} + 1\right) dx$

$= \log_e|1+x| + \log_e|1-x| + x + d = \log_e\left|\frac{1+x}{1-x}\right| + x + d$

When $x = 0$, $y = 4$. $4 = \log_e(1) + 0 + d \rightarrow d = 4$

$\therefore \boxed{y = \log_e\left(\frac{1+x}{1-x}\right) + x + 4}$

e.g. 2. Find the general solution for $\frac{d^2y}{dx^2} = \frac{1}{\cos^2(3x)} = \sec^2(3x)$

$\rightarrow \frac{dy}{dx} = \int \sec^2(3x)\,dx = \frac{1}{3}\tan(3x) + c = \frac{1}{3}\left(\frac{\sin(3x)}{\cos(3x)}\right) + c$

Let $u = \cos(3x)$. $\rightarrow \frac{du}{dx} = -3\sin(3x) \rightarrow du = -3\sin(3x)\,dx$

$\rightarrow y = \int \left(\frac{1}{3}\left(\frac{\sin(3x)}{\cos(3x)}\right) + c\right) dx = -\frac{1}{9}\int \frac{1}{u}\,du + cx + d$

$= -\frac{1}{9}\log_e|u| + cx + d$

$\therefore \boxed{y = -\frac{1}{9}\log_e|\cos(3x)| + cx + d}$

CAS: deSolve(y'' = (sec(3x))^2 and y(0) = -1/8 and y'(0) = 0, x, y) → $y = -\frac{1}{9}\ln(|\cos(3x)|) - \frac{1}{8}$

Exercise 9C Differential equations involving a function of dependent variable

→ Solution of DE's (Case 3)

$\frac{dy}{dx} = g(y) \rightarrow \frac{dx}{dy} = \frac{1}{g(y)} \rightarrow x = \int \frac{1}{g(y)}\,dy = F(y) + c$ (where $F'(y) = \frac{1}{g(y)}$)

e.g. (a) Find the general solution for $\frac{dy}{dx} = 9 - y^2$, $|y| < 3$

$\rightarrow \frac{dx}{dy} = \frac{1}{9-y^2}$, $-3 < y < 3$.

$x = \int \left(\frac{1}{9-y^2}\right) dy = \int \frac{1}{(3-y)(3+y)}\,dy$

$\frac{1}{9-y^2} = \frac{a}{3-y} + \frac{b}{3+y} \rightarrow 1 = a(3+y) + b(3-y)$

Sub $y = -3$, $1 = 6b \rightarrow b = \frac{1}{6}$

Sub $y = 3$, $1 = 6a \rightarrow a = \frac{1}{6}$

$\rightarrow x = \frac{1}{6}\int \left(\frac{1}{3-y} + \frac{1}{3+y}\right) dy$

$= \frac{1}{6}\log_e|3+y| - \frac{1}{6}\log_e|3-y| + c$

$\therefore x = \frac{1}{6}\log_e\left|\frac{3+y}{3-y}\right| + c$

$\rightarrow 6(x-c) = \log_e\left(\frac{3+y}{3-y}\right) \rightarrow \frac{3+y}{3-y} = e^{6(x-c)} = e^{6x} \times e^{-6c}$

Let $A = e^{-6c} \rightarrow \frac{3+y}{3-y} = Ae^{6x}$

$3+y = (3-y)Ae^{6x} \rightarrow y(1+Ae^{6x}) = 3(Ae^{6x}-1)$

$\rightarrow \boxed{y = \frac{3(Ae^{6x}-1)}{Ae^{6x}+1}}$

(b) Hence, or otherwise, find the solution, if $y = 0$ when $x = \frac{7}{6}$.

$\rightarrow \frac{7}{6} = \frac{1}{6}\log_e(1) + c \rightarrow c = \frac{7}{6}$

$\rightarrow 6(x-\frac{7}{6}) = \log_e\left(\frac{3+y}{3-y}\right) \rightarrow e^{6x-7} = \frac{3+y}{3-y}$

$3+y = (3-y)e^{6x-7} \rightarrow y(1+e^{6x-7}) = 3(e^{6x-7}-1)$

$\therefore \boxed{y = \frac{3(e^{6x-7}-1)}{e^{6x-7}+1}}$

Exercise 9D Applications of differential equations

→ Difference of rates": rate of change = rate of increase − rate of decrease

- Steps for set up differential equation

Let m represent amount of salt in tank after t mins

① $\frac{dm}{dt}$ = increase rate − decrease rate (enter − leave)

② # of salt entering tank each min $(L/min) \times (kg/L) \rightarrow (kg/min)$

= (rate of inflow)(concentration of solution in inflow)

③ # of salt leaving tank each min $(L/min) \times (kg/L) \rightarrow (kg/min)$

= (rate of outflow)(concentration of solution in outflow)

④ Concentration of salt in outflow = $\frac{m}{V}$ (kg/L) $C = \frac{Q}{V_0 + \Delta V}$

(V = volume of solution in tank after t mins) $(\Delta V = V_{in} - V_{out})$

Sub expression into Step 3

⑤ Sub Step 2 and 3 into 1

Newton's law of Cooling.

$\frac{dy}{dx} = -k(y-a) \rightarrow y = Ae^{kx} + a \quad (k > 0)$ (where $A = e^{-kc}$)

Tip: concentration = $\frac{\text{quantity}}{\text{volume}}$ or $C = \frac{Q}{V}$

→ Complex examples

e.g. 1. A lab tank contains 100 litres of a 20% serum solution. A 10% serum solution is then pumped in at the rate of 2 litres per minute, and an amount of the solution currently in the tank is drawn off at the same rate.

(a) Set up a differential equation to show the relation between x and t, where x litres is the amount of pure serum in the tank at t mins.

(b) How long will it take for there to be an 18% solution in the tank?

(Assume that contents of tank form a unique solution at all times)

(a) $\frac{dx}{dt} = 2 \times \frac{1}{10} - 2 \times \frac{x}{100} \rightarrow \boxed{\frac{dx}{dt} = \frac{10-x}{50}}$

(b) $\frac{dt}{dx} = \frac{-50}{x-10} \rightarrow t = -50\log_e(x-10) + c$

When $t=0$, $x=20$. $\rightarrow 0 = -50\log_e(10) + c \rightarrow c = 50\log_e(10)$

$\rightarrow t = 50\log_e\left(\frac{10}{x-10}\right)$ → Extra Tip: $x = 10e^{-t/50} + 10$

→ When $x=18$, $t = 50\log_e\left(\frac{5}{4}\right) \approx \boxed{11.16 \text{ mins}}$ (2 dps)

Quick Method: $\frac{dt}{dx} = \frac{-50}{x-10} \rightarrow t = \int_{20}^{18} \frac{-50}{x-10}\,dx = \boxed{11.16 \text{ mins}}$ (2 dps)

e.g. 2. A container of water is heated to boiling point (100°C) and then placed in a room that has a constant temperature of 20°C. After five mins, the temperature of the water is 80°C. [VCAA 2013 Exam 1 Q5]

(a) Use Newton's law of cooling $\frac{dT}{dt} = -k(T-20)$, when T°C is the temperature of the water at time t mins after the water is placed in the room, to show that $e^{-5k} = \frac{3}{4}$.

(b) Find the temperature of the water 10 mins after it is placed inside.

(a) $\frac{dt}{dT} = \frac{-1}{k(T-20)} \rightarrow t = -\frac{1}{k}\int\left(\frac{1}{T-20}\right)dT = -\frac{1}{k}\log_e|T-20| + c$

When $t=0$, $T=100$. $\rightarrow 0 = -\frac{1}{k}\log_e(80) + c \rightarrow c = \frac{1}{k}\log_e(80)$

$\rightarrow t = \frac{1}{k}\log_e\left(\frac{80}{T-20}\right)$

When $t=5$, $T=80 \rightarrow 5 = \frac{1}{k}\log_e\left(\frac{4}{3}\right) \rightarrow 5k = \log_e\left(\frac{4}{3}\right)$

$\rightarrow e^{5k} = \frac{4}{3} \rightarrow \boxed{e^{-5k} = \frac{3}{4}}$ (Shown)

(b) When $t=10$, $10 = \frac{1}{k}\log_e\left(\frac{80}{T-20}\right) \rightarrow 10k = \log_e\left(\frac{80}{T-20}\right)$

$e^{10k} = \frac{80}{T-20} \rightarrow e^{-10k} = \frac{T-20}{80} \rightarrow \left(e^{-5k}\right)^2 = \frac{T-20}{80} = \left(\frac{3}{4}\right)^2$

$T = 80 \times \left(\frac{3}{4}\right)^2 + 20 = \boxed{65°C}$

N.B. Newton's law of cooling: $\frac{dT}{dt} = k(T-T_s)$ $k<0$ | $T = (T_0 - T_s)e^{kt} + T_s$

deSolve($y' = k(y-s), x, y$) → $y = s + ce^{kx}$, $c \in \mathbb{Z}$ (T → variable, T_s → constant)

e.g. 3. VCAA 2013 Exam 2 Q3

Question 3 (11 marks)

The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\log_e(N) = 6 - 3e^{-0.4t}$, $t \geq 0$.

a. Verify by substitution that $\log_e(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation

$$\frac{1}{N}\frac{dN}{dt} + 0.4\log_e(N) - 2.4 = 0.$$

2 marks

→ By implicit differentiation, $\frac{d}{dN}(\log_e(N)) \times \frac{dN}{dt} = \frac{d}{dt}(6 - 3e^{-0.4t})$

$\frac{1}{N}\frac{dN}{dt} = 1.2e^{-0.4t}$ ... ①, $\log_e(N) = 6 - 3e^{-0.4t}$ ... ②

Sub ① and ② into differential equation,

LHS $= 1.2e^{-0.4t} + 0.4(6 - 3e^{-0.4t}) - 2.4$

$= 1.2e^{-0.4t} + 2.4 - 1.2e^{-0.4t} - 2.4$

$= \boxed{0} =$ RHS (Shown)

b. Find the initial number of phones owned in the community.
Give your answer correct to the nearest integer.

1 mark

When $t = 0$, $\log_e(N) = 3 \rightarrow N = e^3 \rightarrow N = \boxed{20}$

c. Using this mathematical model, find the limiting number of mobile phones that would eventually be owned in the community.
Give your answer correct to the nearest integer.

2 marks

When $t \to \infty$, $\log_e(N) \to 6$ $\therefore N \to e^6 \approx \boxed{403}$

The differential equation in **part a.** can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \log_e(N))$.

d. **i.** Find $\frac{d^2N}{dt^2}$ in terms of N and $\log_e(N)$.

2 marks

→ $\frac{d^2N}{dt^2} = \frac{d}{dN}\left(\frac{dN}{dt}\right) \times \frac{dN}{dt} = [0.4 \times (6 - \log_e(N)) + 0.4N \times -\frac{1}{N}] \times \frac{dN}{dt}$

$= (2 - 0.4\log_e(N))(2.4N - 0.4N\log_e(N))$

$\therefore = \boxed{0.16N(6 - \log_e(N))(5 - \log_e(N))}$

ii. The graph of N as a function of t has a point of inflection.
Find the values of the coordinates of this point.
Give the value of t correct to one decimal place and the value of N correct to the nearest integer.

2 marks

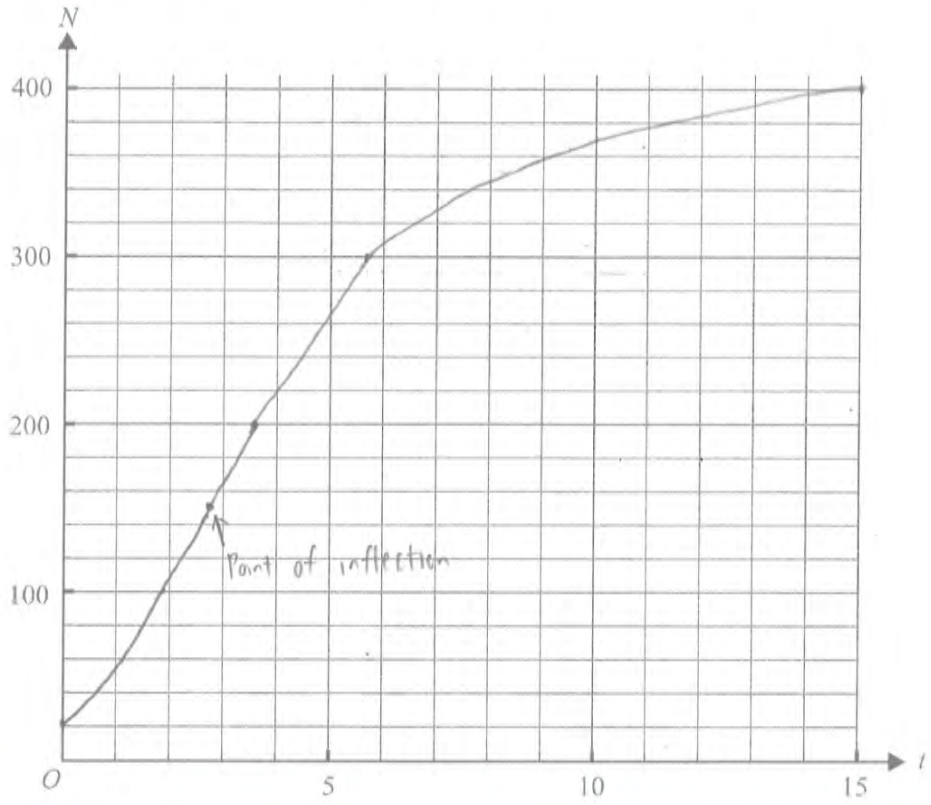
→ $\frac{d^2N}{dt^2} = 0 \rightarrow 5 - \log_e(N) = 0$

→ $N = e^5 \approx 148$

Solve $\log_e(e^5) = 6 - 3e^{-0.4t}$ for t, $t = 2.7$

→ Coordinates: $\boxed{(2.7, 148)}$

e. Sketch the graph of N as a function of t on the axes below for $0 \leq t \leq 15$. 2 marks



Exercise 9E Separation of variables

→ Solution of DE's (Case 4)

· $\frac{dy}{dx} = f(x)\,g(y) \rightarrow \int f(x)\,dx = \int \frac{1}{g(y)}\,dy$

→ Complex examples

· e.g. 1. Find the general solution of differential equation $\frac{dy}{dx} = y(1+e^x)$

→ $\frac{1}{y}\left(\frac{dy}{dx}\right) = 1+e^x \rightarrow \int \frac{1}{y}\,dy = \int (1+e^x)\,dx$

$\log_e|y| = x + e^x + c$

$|y| = e^{x+e^x+c} = e^{x+e^x} \times e^c$ (Let $A = e^c \rightarrow$ a constant)

→ $\boxed{y = Ae^{x+e^x}}$

· e.g. 2. Solve the differential equation $\sqrt{2-x^2}\left(\frac{dy}{dx}\right) = \frac{1}{2-y}$, given than $y(1) = 0$.

Express y as a function of x (VCAA 2016 Exam 1 Q10)

→ $2-y = \frac{1}{\sqrt{2-x^2}}\left(\frac{dx}{dy}\right) \rightarrow \int (2-y)\,dy = \int \left(\frac{1}{\sqrt{2-x^2}}\right) dx$

$2y - \frac{1}{2}y^2 = \arcsin\left(\frac{x}{\sqrt{2}}\right) + c$

When $x = 1$, $y = 0 \rightarrow -0 = \arcsin\left(\frac{1}{\sqrt{2}}\right) + c = \frac{\pi}{4} + c \rightarrow c = -\frac{\pi}{4}$

$\rightarrow -2y - \frac{1}{2}y^2 = \arcsin\left(\frac{\sqrt{2}}{2}x\right) - \frac{\pi}{4}$

$y^2 - 4y = -2\arcsin\left(\frac{\sqrt{2}}{2}x\right) + \frac{\pi}{2}$

$(y^2 - 4y + 4) - 4 = -2\arcsin\left(\frac{\sqrt{2}}{2}x\right) + \frac{\pi}{2}$

$(y-2)^2 = 4 + \frac{\pi}{2} - 2\arcsin\left(\frac{\sqrt{2}}{2}x\right)$

$y = 2 \pm \sqrt{4 + \frac{\pi}{2} - 2\arcsin\left(\frac{\sqrt{2}}{2}x\right)}$

When $x = 1$, $y = 0$.

$\rightarrow \boxed{y = 2 - \sqrt{4 + \frac{\pi}{2} - 2\arcsin\left(\frac{\sqrt{2}}{2}x\right)}}$

e.g. 3. VCAA 2016 ~~Exam 2~~ Q3

Question 3 (11 marks)

A tank initially has 20 kg of salt dissolved in 100 L of water. Pure water flows into the tank at a rate of 10 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 5 L/min.

If x kilograms is the amount of salt in the tank after t minutes, it can be shown that the differential equation relating x and t is $\frac{dx}{dt} + \frac{x}{20+t} = 0$.

a. Solve this differential equation to find x in terms of t. 3 marks

$\rightarrow \frac{dx}{dt} = -\frac{x}{20+t} \rightarrow \int \frac{1}{x}\,dx = \int \frac{-1}{20+t}\,dt$ (AS: deSolve $\left(x' = \frac{-x}{20+t}\right.$ and $\left.x(0) = 20, t, x\right)$

$\log_e(x) = -\log_e(20+t) + c$

When $t = 0$, $x = 20$. $\rightarrow \log_e(20) = -\log_e(20) + c \rightarrow c = 2\log_e(20) = \log_e(400)$

$\rightarrow \log_e(x) = \log_e\left(\frac{400}{20+t}\right) \rightarrow \boxed{x = \frac{400}{20+t}}$

A second tank initially has 15 kg of salt dissolved in 100 L of water. A solution of $\frac{1}{60}$ kg of salt per litre flows into the tank at a rate of 20 L/min. The solution of salt and water, which is kept uniform by stirring, flows out of the tank at a rate of 10 L/min.

b. If y kilograms is the amount of salt in the tank after t minutes, write down an expression for the **concentration**, in kg/L, of salt in the second tank at time t. 1 mark

$\rightarrow \boxed{\frac{y}{100+10t} \text{ kg/L}}$ (since volume $= 100 + (20-10)t$)

c. Show that the differential equation relating y and t is $\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}$. 2 marks

$\frac{dy}{dt}$ (in): Rate of inflow: $20 \times \frac{1}{60} = \frac{1}{3}$ kg/min

$\frac{dy}{dt}$ (out): Rate of outflow: $10 \times \frac{y}{100+10t} = \frac{y}{10+t}$ kg/min

$\rightarrow \frac{dy}{dt} = \frac{1}{3} - \frac{y}{10+t} \rightarrow \boxed{\frac{dy}{dt} + \frac{y}{10+t} = \frac{1}{3}}$ (Shown)

N.B. concentration $= \frac{\text{amount}}{\text{volume}}$

d. Verify by differentiation and substitution into the left side that $y = \dfrac{t^2 + 20t + 900}{6(10 + t)}$ satisfies the **differential equation** in **part c.** Verify that the given solution for y also satisfies the **initial condition**. 3 marks

$\rightarrow \frac{y}{10+t} = \frac{t^2+20t+900}{6(10+t)^2}$ ... ①

$\rightarrow \frac{dy}{dt} = \frac{1}{6} \times \frac{(10+t)(2t+20) - (t^2+20t+900)}{(10+t)^2} = \frac{t^2+20t-700}{6(10+t)^2}$ ... ②

Sub ① and ② into equation,

LHS $= \frac{t^2+20t+900}{6(10+t)^2} + \frac{t^2+20t-700}{6(10+t)^2} = \frac{2t^2+40t+200}{6(10+t)^2}$

$= \frac{2(10+t)^2}{6(10+t)^2} = \boxed{\frac{1}{3}} =$ RHS (Shown)

When $t = 0$, $y = \frac{900}{6 \times 10} = \boxed{15}$ (Shown)

e. Find when the **concentration** of salt in the second tank reaches 0.095 kg/L. Give your answer in minutes, correct to two decimal places. 2 marks

$\rightarrow \frac{y}{100+10t} = \frac{y}{10(10+t)} = \frac{t^2+20t+900}{60(10+t)^2} = 0.095$

Solve for t, $t \approx -23.05$ (reject) or $t = 3.05$ (accept)

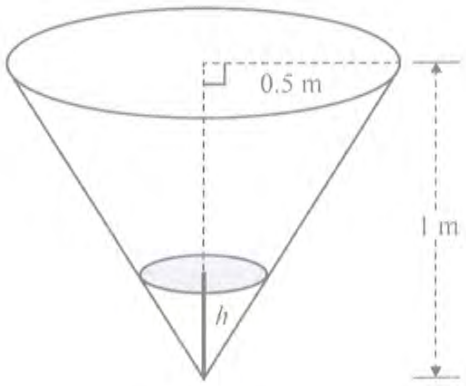
$\rightarrow t \approx \boxed{3.05 \text{ mins}}$ (2 dps)

Exercise 9F Differential equations with related rates

e.g. 1. VCAA 2014 Exam 2 Q4

Question 4 (12 marks)

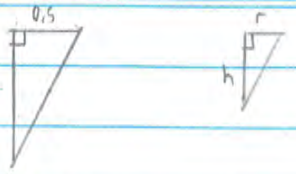
At a water fun park, a conical tank of radius 0.5 m and height 1 m is filling with water. At the same time, some water flows out from the vertex, wetting those underneath. When the tank eventually fills, it tips over and the water falls out, drenching all those underneath. The tank then returns to its original position and begins to refill.



Water flows in at a constant rate of 0.02π m^3/min and flows out at a variable rate of $0.01\pi\sqrt{h}$ m^3/min, where h metres is the depth of the water at any instant.

a. Show that the volume, V cubic metres, of water in the cone when it is filled to a depth of h metres is given by $V = \dfrac{\pi}{12}h^3$. 1 mark

(a) By similar triangles, $\frac{r}{h} = \frac{0.5}{1} = \frac{1}{2} \rightarrow r = \frac{h}{2}$



$\rightarrow V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \boxed{\frac{\pi}{12}h^3}$ (Shown)

b. Find the rate, in m/min, at which the depth of the water in the tank is increasing when the depth is 0.25 m. 4 marks

$\rightarrow \frac{dV}{dt} = 0.02\pi - 0.01\pi\sqrt{h}$

$V = \frac{\pi}{12}h^3 \rightarrow \frac{dV}{dh} = \frac{\pi}{4}h^2 \rightarrow \frac{dh}{dV} = \frac{4}{\pi h^2}$

$\rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{4}{\pi h^2}(0.02\pi - 0.01\pi\sqrt{h}) = \frac{2-\sqrt{h}}{25h^2}$

Sub $h = \frac{1}{4}$, $\frac{dh}{dt} = \frac{2-\frac{1}{2}}{25(\frac{1}{16})} = \boxed{0.96 \text{ m/min}}$

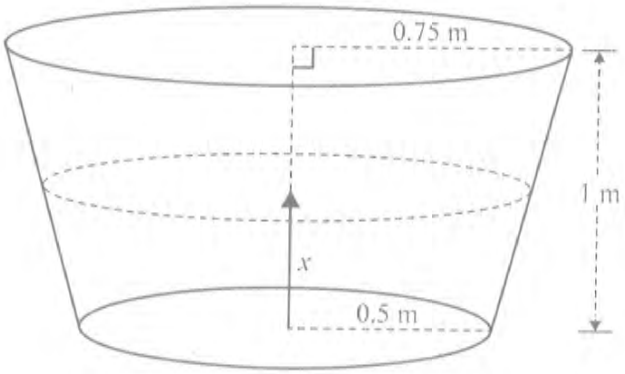
(c) The tank is empty at time $t = 0$ mins. (2 marks)

By using an appropriate definite integral, find the time it takes for the tank to fill. Give your answer in minutes, correct to one decimal place.

$\rightarrow \frac{dh}{dt} = \frac{2-\sqrt{h}}{25h^2} \rightarrow \frac{dt}{dh} = \frac{25h^2}{2-\sqrt{h}}$

$t = \int_0^1 \left(\frac{25h^2}{2-\sqrt{h}}\right) dh \approx \boxed{7.4 \text{ minutes}}$ (1.d.p)

Another water tank, shown below, has the shape of a large bucket (part of a cone) with the dimensions given. Water fills the tank at a rate of 0.05π m^3/min, but no water leaks out.



When filled to a depth of x metres, the volume of water, V cubic metres, in the tank is given by

$$V = \frac{\pi}{48}\left(x^3 + 6x^2 + 12x\right)$$

d. Given that the tank is initially empty, find the depth, x metres, as a function of time t. 5 marks

$\rightarrow \frac{dV}{dt} = \frac{\pi}{20}$, $\frac{dV}{dx} = \frac{\pi}{48}(3x^2 + 12x + 12) = \frac{\pi}{16}(x+2)^2 \rightarrow \frac{dx}{dV} = \frac{16}{\pi(x+2)^2}$

$\rightarrow \frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt} = \frac{16}{\pi(x+2)^2} \times \frac{\pi}{20} = \frac{4}{5(x+2)^2}$

$\frac{dt}{dx} = \frac{5(x+2)^2}{4} \rightarrow t = \frac{5}{4}\int (x+2)^2 dx = \frac{5}{12}(x+2)^3 + C$

Sub $t = 0$, $x = 0$, $0 = \frac{5}{12}(8) + C \rightarrow C = -\frac{10}{3} \rightarrow t = \frac{5}{12}(x+2)^3 - \frac{10}{3}$

$\rightarrow \frac{12}{5}\left(t + \frac{10}{3}\right) = (x+2)^3 \rightarrow \boxed{x = \left(\frac{12}{5}\left(t + \frac{10}{3}\right)\right)^{1/3} - 2}$

e.g. 2. A vessel in the shape of a right circular cone has a vertical axis and a semi-vertex angle of 30°. There is a small hole at the vertex so that liquid leaks out at the rate of $0.05\sqrt{h}\ m^3$ per hour, where h metres is the depth of liquid in the vessel at time t hours. Given that the liquid is poured into this vessel at a constant rate of $2m^3$ per hour, set up a differential equation for h.

$\rightarrow r = h\tan(30^\circ) = \frac{\sqrt{3}h}{3}$

$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{\sqrt{3}h}{3}\right)^2 h = \frac{\pi h^3}{9}$

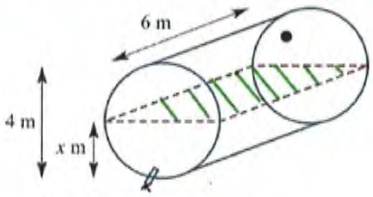
$\rightarrow \frac{dV}{dh} = \frac{\pi h^2}{3}, \quad \frac{dh}{dV} = \frac{3}{\pi h^2}, \quad \frac{dV}{dt} = 2 - 0.05\sqrt{h}$

$\rightarrow \frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} = \frac{3}{\pi h^2}\left(2 - 0.05\sqrt{h}\right)$

$\frac{dh}{dt} = \boxed{\frac{6 - 0.15\sqrt{h}}{\pi h^2}}$

e.g. 3.

3 A cylindrical tank is lying on its side, as shown in the figure. The tank has a hole in the top, and another in the bottom so that the water in the tank leaks out. If the depth of water is x m at time t minutes and $\frac{dx}{dt} = \frac{-0.025\sqrt{x}}{A}$ where A m^2 is the surface area of the water at t minutes:



a construct the differential equation expressing $\frac{dx}{dt}$ as a function of x only

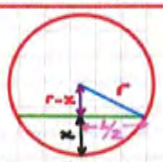
b solve the differential equation given that initially the tank was full

c find how long it will take to empty the tank

(a) Now $\frac{dx}{dt} = \frac{-0.025\sqrt{x}}{A}$

However, A is not constant.
A depends on x.

$A = L \times 6$ where 6 is a constant.
L is affected by x.



From this diagram, by Pythagoras' theorem,

$\frac{L}{2} = \sqrt{r^2 - (r-x)^2} = \sqrt{r^2 - (r^2 - 2xr + x^2)}$

$= \sqrt{2xr - x^2}$

So $L = 2\sqrt{2xr - x^2}$

Sub $r = 2$

$\rightarrow \boxed{L = 2\sqrt{4x - x^2}}$

So $A = 6L$

$= 12\sqrt{4x - x^2}$

$\rightarrow \frac{dx}{dt} = \frac{-0.025 \times \sqrt{x}}{12\sqrt{4x - x^2}}$

$\frac{dx}{dt} = \frac{-0.025 \times \sqrt{x}}{12\sqrt{4x - x^2}}$ (recap)

$\rightarrow \boxed{\frac{dx}{dt} = \frac{-1}{480\sqrt{4x - x^2}}}$

(b) $\rightarrow \frac{dt}{dx} = -480\sqrt{4-x}$

$\rightarrow t = -480\int (4-x)^{1/2}\,dx$

Let $u = 4 - x \rightarrow \frac{du}{dx} = -1$

$\rightarrow du = -dx$

So we have

$t = -480\int u^{1/2} \cdot -du$

$= +480\int u^{1/2} \cdot du$

$= 480 \times \frac{2}{3}u^{3/2} + C$

Sub back for u:—

$\rightarrow t = 320(4-x)^{3/2} + C$

Sub $x = 4, t = 0 \rightarrow C = 0$

$\rightarrow \boxed{t = 320(4-x)^{3/2}}$

Solving for x:—

$\rightarrow (4-x)^{3/2} = \frac{t}{320}$

$\rightarrow 4 - x = \left(\frac{t}{320}\right)^{2/3}$

$\rightarrow \boxed{x = 4 - \left(\frac{t}{320}\right)^{2/3}}$

(c) $t = 320(4-x)^{3/2}$

Sub $x = 0$ (empty)

$\rightarrow t = 320 \times 4^{3/2}$

$= 320 \times 8$

$= 2560$ min

$= \boxed{42 \text{ hours}, 40 \text{ min}}$

Torcelli's Law: $\frac{dh}{dt} = \frac{-h\sqrt{h}}{A}$

Exercise 9G Using a definite integral to solve a differential equation

Using a definite integral, solve the differential equation $\frac{dy}{dx} = f(x)$ at $x = b$, given than $y = d$ when $x = a$.

$\rightarrow y(b) - y(a) = \int_a^b f(x)\,dx$

$y(b) = y(a) + \int_a^b f(x)\,dx$

$= \boxed{d + \int_a^b f(x)\,dx}$

e.g. $\frac{dy}{dx} = \log_e(x^2)$, $y = 2$ when $x = 1$

$y(e) - y(1) = \int_1^e \log_e(x^2)\,dx$

$y(e) = y(1) + \int_1^e \log_e(x^2)\,dx$

$= 2 + [x\ln(x^2) - 2x]_1^e$ (integration by recognition)

$= 2 + e\ln(e^2) - 2e - 0 + 2$

$= \boxed{4}$

Exercise 9H Euler's Method for solving differential equations

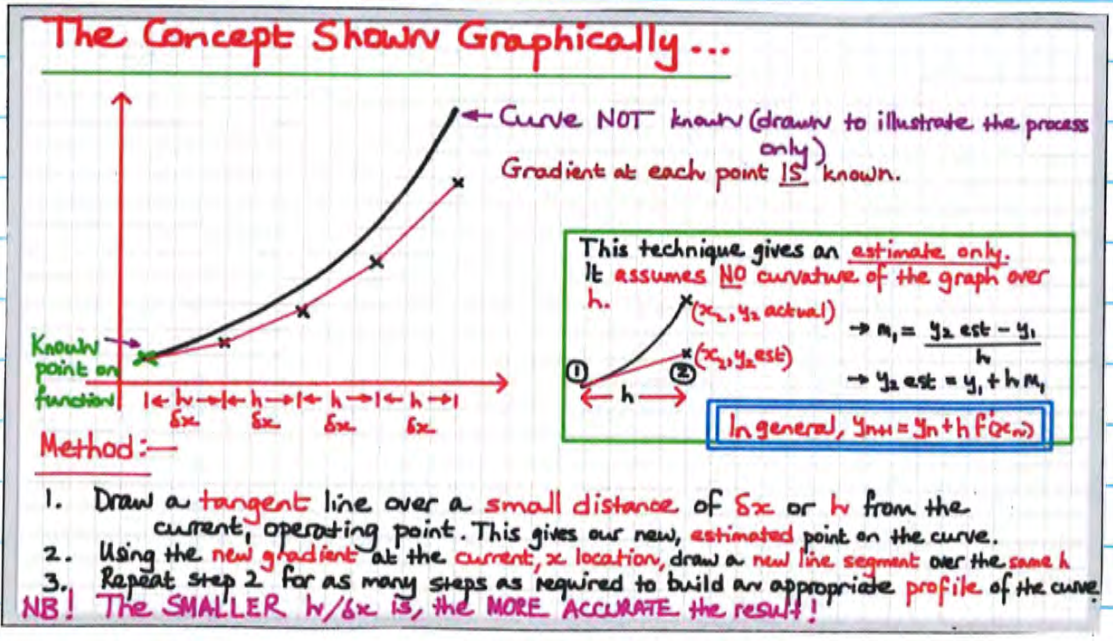
→ Euler's Method (numerical method for calculating approximate values of solution of $\frac{dy}{dx} = f(x, y)$ given $y(x_0) = y_0$) $\quad \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$

- used if functions can't be integrated analytically
- determined with small increments of x
- gradient assumed constant → approximate corresponding next value of y

• $x_{n+1} = x_n + h$ ∴ $y(x_{n+1}) \approx y(x_n) + hf(y_n, x_n)$

from linear approximation $f(x+h) \approx f(x) + hf'(x)$

The Concept Shown Graphically ...



Method:—

1. Draw a tangent line over a small distance of δx or h from the current, operating point. This gives our new, estimated point on the curve.
2. Using the new gradient at the current, x location, draw a new line segment over the same h
3. Repeat step 2 for as many steps as required to build an appropriate profile of the curve.

NB! The SMALLER $h/\delta x$ is, the MORE ACCURATE the result!

An Example, using Tabular Format for Computation.

Assume we cannot find $\int_0^1 2x\,dx$ where $x_0 = 0$, $y_0 = 1$, and we require y when $x = 1$.

So $\frac{dy}{dx} = 2x$

CASE ① :– Let $h = 0.2$

n	x_n	y_n	$\frac{dy}{dx}\Big\|_{x=x_n}$	$h \times \frac{dy}{dx}\Big\|_{x_n} = \delta y$	$y_n + \delta y = y_{n+1}$
0	0	1	$2\times0 = 0$	$0.2\times0 = 0$	$1+0 = 1$
1	0.2	1	$2\times0.2 = 0.4$	$0.2\times0.4 = 0.08$	$1+0.08 = 1.08$
2	0.4	1.08	$2\times0.4 = 0.8$	$0.2\times0.8 = 0.16$	$1.08+0.16 = 1.24$
3	0.6	1.24	$2\times0.6 = 1.2$	$0.2\times1.2 = 0.24$	$1.24+0.24 = 1.48$
4	0.8	1.48	$2\times0.8 = 1.6$	$0.2\times1.6 = 0.32$	$1.48+0.32 = 1.80$
5	1.0	1.80			

So when $x = 1$, $y \simeq 1.80$

Now when $x = 1$,

$$y_{actual} = 1 + \int_0^1 2x\,dx$$
$$= 1 + \left[x^2\right]_0^1$$
$$= 1 + 1$$
$$= 2$$

This was NOT very accurate!

To increase the accuracy we need to make h SMALLER and proceed in SMALLER steps ...

CASE ② :– Let $h = 0.1$

n	x_n	y_n	$\frac{dy}{dx}\Big\|_{x=x_n}$	$h \times \frac{dy}{dx}\Big\|_{x_n} = \delta y$	$y_n + \delta y = y_{n+1}$
0	0	1	0	$0.1\times0= 0$	$1+0 = 1$
1	0.1	1	$2\times0.1 = 0.2$	$0.1\times0.2 = 0.02$	$=1+0.02 = 1.02$
2	0.2	1.02	$2\times0.2 = 0.4$	$0.1\times0.4 = 0.04$	$1.02+0.04 = 1.06$
3	0.3	1.06	$2\times0.3 = 0.6$	$0.1\times0.6 = 0.06$	$1.06+0.06 = 1.12$
4	0.4	1.12	$2\times0.4 = 0.8$	$0.1\times0.8 = 0.08$	$1.12+0.08 = 1.20$
5	0.5	1.20	$2\times0.5 = 1.0$	$0.1\times1.0 = 0.10$	$1.20+0.10 = 1.30$
6	0.6	1.30	$2\times0.6 = 1.2$	$0.1\times1.2 = 0.12$	$1.30+0.12 = 1.42$
7	0.7	1.42	$2\times0.7 = 1.4$	$0.1\times1.4 = 0.14$	$1.42+0.14 = 1.56$
8	0.8	1.56	$2\times0.8 = 1.6$	$0.1\times1.6 = 0.16$	$1.56+0.16 = 1.72$
9	0.9	1.72	$2\times0.9 = 1.8$	$0.1\times1.8 = 0.18$	$1.72+0.18 = 1.90$
10	1.0	1.90			

So $y_{actual} = 2$

y_{est} is 1.90 when $h = 0.1$

y_{est} is 1.80 when $h = 0.2$

Euler's method can be done on CAS :–

euler(2·x,x,y,{0,1},1,0.2)

Select 'Euler' from Catalogue :–
Type 'e' to gain access to entries starting with 'e'.

$$\begin{bmatrix} 0. & 0.2 & 0.4 & 0.6 & 0.8 & 1. \\ 1. & 1. & 1.08 & 1.24 & 1.48 & 1.8 \end{bmatrix}$$

$\frac{dy}{dx} = f$

Independent variable, x

Dependent variable, y

{Starting x, finishing x} must use curley brackets

Starting value of y

Step size, h

euler(Expr, Var, depVar, {VarMin, VarMax}, depVar0, VarStep, eulerStep)

euler($\frac{dy}{dx}$, x, y, {x_0, x_n}, y_0, h)

As h↓, accuracy ↑ :–

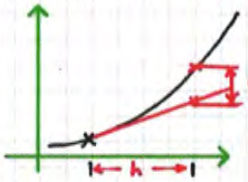
euler(2·x,x,y,{0,1},1,0.01) h = 0.01 100 calculations Inaccurate by 0.01

.9	0.9	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.
32	1.801	1.819	1.8372	1.8556	1.8742	1.893	1.912	1.9312	1.9506	1.9702	1.99

Calc command: euler($\frac{dy}{dx}$, x, y, {x_0, x_1}, y_0, step size)

Over-estimate or Under-estimate?

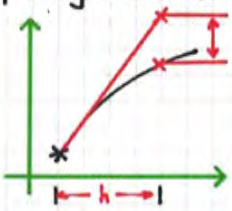
If the gradient is increasing as x increases → under-estimate.



concave up

$\frac{d^2y}{dx^2} > 0$

If the gradient is decreasing as x increases → over-estimate.



concave down

$\frac{d^2y}{dx^2} < 0$

e.g. VCAA 2007 Exam 1 Q7

Question 7

a. Use Euler's method to find y_2 if $\frac{dy}{dx} = \frac{1}{x}$, given that $y_0 = y(1) = 1$ and $h = 0.1$.

Express your answer as a fraction.

2007 Exam 1

$x_0 = 1,\ y_0 = 1 \quad \left.\frac{dy}{dx}\right|_{x=1} = 1 \rightarrow h \times \left.\frac{dy}{dx}\right|_{x=1} = 0.1 \rightarrow y_1 = 1 + 0.1 = 1.1$

$x_1 = 1.1,\ y_1 = 1.1 \quad \left.\frac{dy}{dx}\right|_{x=1.1} = \frac{1}{1.1} \rightarrow h \times \left.\frac{dy}{dx}\right|_{x=1.1} = 0.1 \times \frac{1}{1.1} \rightarrow y_2 = 1.1 + \frac{0.1}{1.1}$

As a fraction, $1.1 + \frac{0.1}{1.1} = \frac{1.1 \times 1.1 + 0.1}{1.1} = \frac{1.21 + 0.1}{1.1} = \frac{1.31}{1.1} = \boxed{\frac{131}{110}}$

b. Solve the differential equation given in part a. to find the value of y which is estimated by y_2.

Express your answer in the form $\log_e(a) + b$, where a and b are positive real constants.

$\frac{dy}{dx} = \frac{1}{x} \rightarrow y = \log_e|x| + c$ Sub $x = 1, y = 1 \rightarrow 1 = \log_e(1) + c \rightarrow c = 1 \rightarrow \boxed{y = \log_e|x| + 1}$

So when $x = 1.2$, $\boxed{y = \log_e(1.2) + 1}$

e.g. VCAA 2017 Exam 2 Q9

Consider $\frac{dy}{dx} = 2x^2 + x + 1$, where $y(1) = y_0 = 2$. Using Euler's method with a step size of 0.1, find an approximation to $y(0.8) = y_2$

$x_0 = 1$	$y_0 = 2$	$\frac{dy}{dx} = 4$
$x_1 = 0.9$	$y_1 = 2 - 0.1 \times 4 = 1.6$	$\frac{dy}{dx} = 3.52$
$x_2 = 0.8$	$y_2 = 1.6 - 0.1 \times 3.52 = \boxed{1.248}$	

$\text{euler}(2x^2 + x + 1, x, y, \{1, 0.8\}, 2, -0.1)$

e.g. Consider $\frac{dx}{dt} = \sqrt{\ln(x+2)} + 2$, $x > -1$ and $x = 0$ when $t = 1$

(a) Use Euler's Method with a step size of 0.2 to find value of x when $t = 2$, correct to four decimal places.

(b) Explain without comparing with the 'exact answer' why the answer above is an overestimate or an underestimate of the exact value of x when $t = 2$.

(a) $\text{Euler}(\sqrt{\ln(x+2)} + 2, t, x, \{1, 2\}, 0, 0.2)$

$\rightarrow x = 3.0493$

(b) $\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt}\left(\sqrt{\ln(x+2)} + 2\right) = \frac{d}{dx}\left(\sqrt{\ln(x+2)} + 2\right) \times \frac{dx}{dt}$

$= \frac{1}{2(x+2)\sqrt{\ln(x+2)}} \times \left(\sqrt{\ln(x+2)} + 2\right)$

$\frac{d^2x}{dt^2} > 0$ for $x > -1$

$\therefore$ For $t \in [1, 2]$, there will be values of x such that $x \in (-1, \infty)$.

Therefore $\frac{d^2x}{dt^2} > 0$ for $t \in [1, 2]$.

Exercise 9I Slope fields for Differential Equations

→ Slope field → $\frac{dy}{dx} = f(x, y)$ assigns to each point $P = (x, y)$ in plane (for which $f(x, y)$ is defined) the number $f(x, y)$

- gradient of solution curve through P
- direction field determines concavity of solution curve → Euler's method overestimates or underestimates exact value of y

What are Slope Fields ?

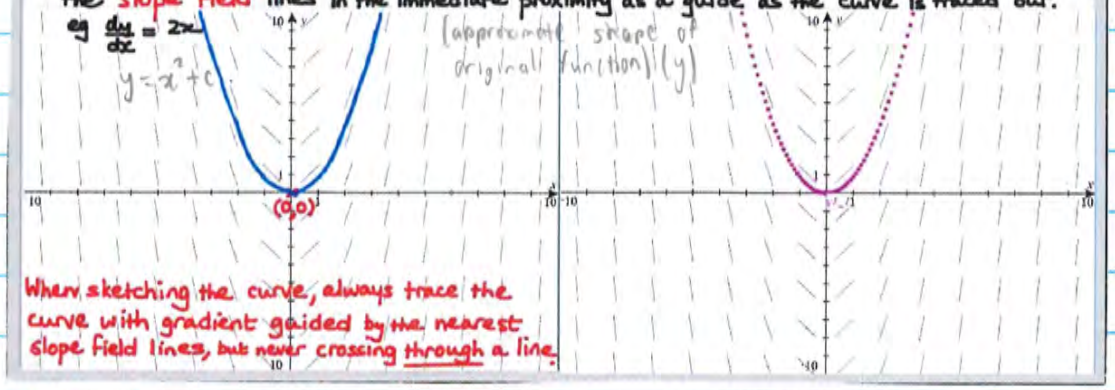
They are another method of finding a graph from a known derivative function.

A two-dimensional array of x and y positions on a cartesian plane is defined, and then, mini-gradient lines are drawn at each of these points over a 2-dimensional array.

Then, if a single, known point on the curve is known, then the curve can be sketched using the slope field lines in the immediate proximity as a guide as the curve is traced out.

eg $\frac{dy}{dx} = 2x$ (approximate shape of original function) (y)

$y = x^2 + c$



When sketching the curve, always trace the curve with gradient guided by the nearest slope field lines, but never crossing through a line.

e.g. VCAA 2007 Exam 1 Q8

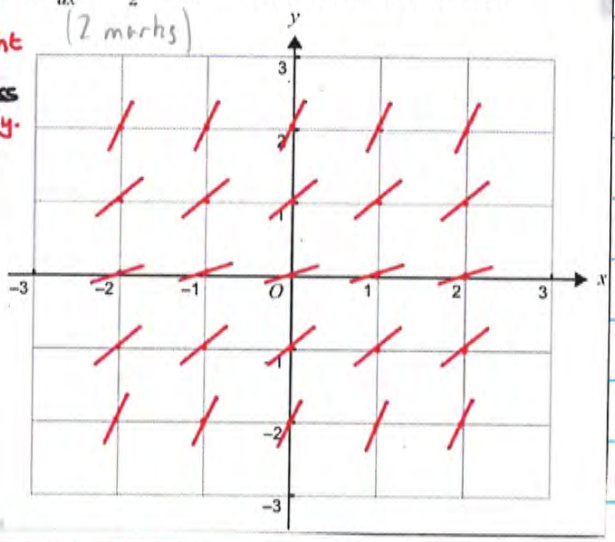
Question 8

a. Sketch the slope field of the differential equation $\frac{dy}{dx} = \frac{1+y^2}{2}$ for $y = -2, -1, 0, 1, 2$ at each of the values $x = -2, -1, 0, 1, 2$ on the axes below.

(2 marks)

$\frac{dy}{dx}$ depends only on y. So the gradient lines will be the same horizontally across the graph, but they will change vertically.

y	-2	-1	0	1	2
$\frac{dy}{dx}$	5/2	1	1/2	1	5/2



b. If $y = -1$ when $x = 0$, solve the differential equation given in **part a.** to find y in terms of x.

(3 marks)

$\frac{dy}{dx} = \frac{1+y^2}{2}$

$\rightarrow \frac{dx}{dy} = \frac{2}{1+y^2}$

$\rightarrow x = 2\tan^{-1}(y) + c$

Sub $x=0, y=-1$

$\rightarrow 0 = 2\times\tan^{-1}(-1) + c$

$\rightarrow 0 = 2\times(-\pi/4) + c$

$\rightarrow 0 = -\pi/2 + c$

$\rightarrow c = \pi/2$

Hence, $x = 2\tan^{-1}(y) + \pi/2$

So $\tan^{-1}(y) = \frac{1}{2}(x - \pi/2)$

$\rightarrow$ $y = \tan\frac{1}{2}(x - \pi/2)$

c. Sketch the graph of the solution curve found in **part b.** on the slope field in **part a.**

(1 mark)

$y = \tan\frac{1}{2}(x - \pi/2)$

When $x=0$, $y=-1$

Need x-intercept :–

When $y=0$, $\tan\frac{1}{2}(x-\pi/2) = 0$

$\rightarrow \frac{1}{2}(x-\pi/2) = 0 \rightarrow x = \pi/2 \sim \frac{3.14}{2} \sim 1.6$

Asymptotes:–

$\frac{1}{2}(x-\pi/2) = \pi/2 + k\pi,\ k\in\mathbb{Z}$

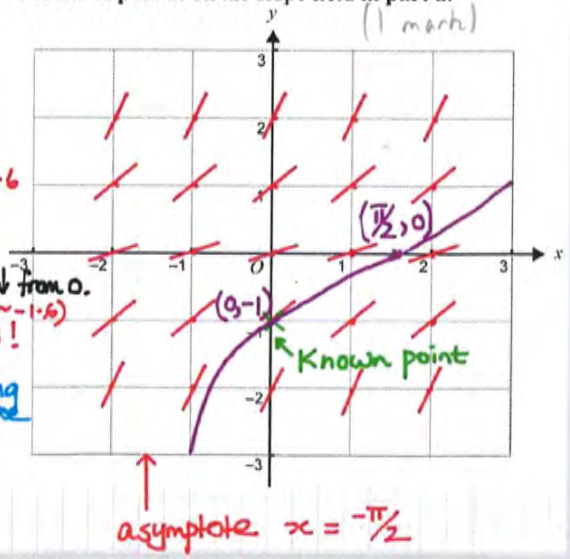
$\rightarrow x - \pi/2 = \pi + 2k\pi$

$\rightarrow x = 3\pi/2 + 2k\pi$ ie $3\pi/2, -\pi/2, -5\pi/2$ as k↓ from 0.

Hence there IS an asymptote at $x = -\pi/2$ (~ -1.6)

We have to allow for any influence of this!

We can now sketch the graph, knowing enough about it to have full confidence in what we are doing!



VCAA 2009 Exam 2 Q9

The direction field shown above could be that of the differential equation

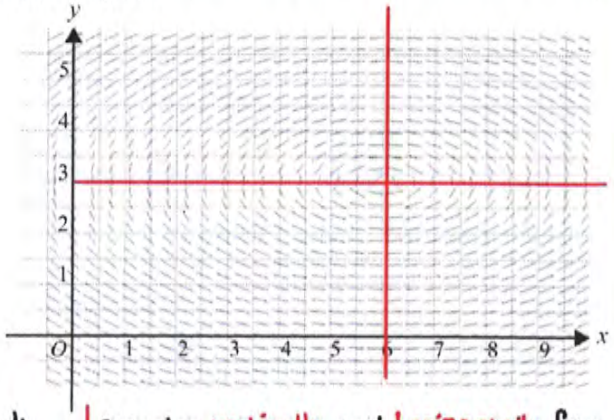
A. $\frac{dy}{dx} = \frac{(x-6)^2}{36} + \frac{(y-3)^2}{9}$

B. $\frac{dy}{dx} = \frac{6-x}{4(y-3)}$

C. $\frac{dy}{dx} = \frac{6+x}{4(y+3)}$

D. $\frac{dy}{dx} = \frac{6-x}{4(y+3)}$

E. $\frac{dy}{dx} = \frac{(x+6)^2}{36} + \frac{(y+3)^2}{9}$



This looks like the shape of an ellipse. However, we are asked for the equation of $\frac{dy}{dx}$, not y!

In this example, the x and y axes don't help much, because the gradient varies through -ve, zero and positive values for both.

However, from the centre of the ellipse at around (6,3) we can see that by scanning vertically and horizontally from this point, that when $x=6$, $\frac{dy}{dx}=0$. When $y=3$, $\frac{dy}{dx}$ is undefined (vertical lines).

(B) is the only option which matches this profile. Hence B

VCAA 2011 Exam 2 Q17

The differential equation which best represents the above direction field is

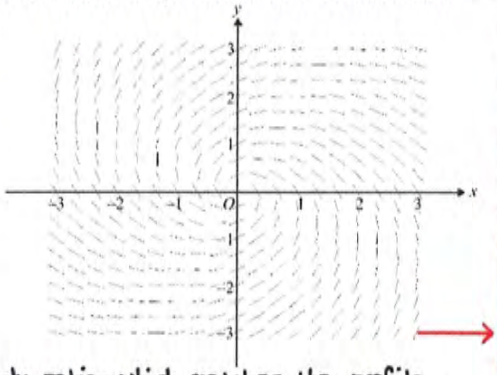
A. $\frac{dy}{dx} = \frac{y-2x}{2y+x}$

B. $\frac{dy}{dx} = \frac{2x-y}{y-2x}$

C. $\frac{dy}{dx} = \frac{2y-x}{y+2x}$

D. $\frac{dy}{dx} = \frac{y-2x}{2y-x}$

E. $\frac{dy}{dx} = \frac{x-2y}{2y+x}$



(A) is the only option which matches the profile described above.

→ A

Scanning the x and y axes,

When $x=0$ (y axis), $\frac{dy}{dx}$ is slightly (somewhat) +ve.

When $y=0$ (x-axis), $\frac{dy}{dx}$ is quite negative (more -ve than the degree to which it is +ve when $x=0$ above).

Analysing the 5 options:-

Option $\frac{dy}{dx}$	dy/dx when $x=0$ (y-axis)	$y=0$ (x-axis)
A $\frac{y-2x}{2y+x}$	$\frac{1}{2}$	-2
B $\frac{2y-x}{y-2x}$	2	$\frac{1}{2}$
C $\frac{2y-x}{y+2x}$	2	$-\frac{1}{2}$
D $\frac{y-2x}{2y-x}$	$\frac{1}{2}$	+2
E $\frac{x-2y}{2y+x}$	-1	+1

→ CAS skills (Graphs)

- Menu → Graph Entry/Edit (3) → Diff Eq (8)
- e.g.s: $y1' = \frac{y1-2x}{2y1+x}$, $y1' = y1^2 - x$

N.B. Set window to zoom fit

Can't plot 2 DEs per Q (only 1 DE per page)

If $\frac{dy}{dx}$ has y values, make y into 'y1'

VCAA 2008 Exam 2 Q 4 (Ex 9D)

An island has a population of rabbits and a population of foxes. The foxes eat rabbits as their food source and if the rabbit population decreases, then after some time, so will the fox population. Also, if the rabbit population increases, then after some time, so too will the fox population.

At time t months from the start of the year there are x thousand rabbits and y hundred foxes. A model for the two populations is given by the parametric equations

Rabbits $x = 10 + 3\cos\left(\frac{\pi t}{6}\right), t \geq 0$

Foxes $y = 5 + \sin\left(\frac{\pi t}{6}\right), t \geq 0.$

a. Find the cartesian equation relating x and y according to this model.

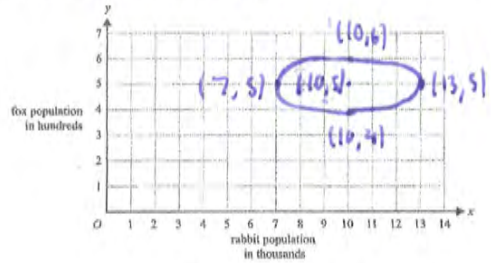
$\cos\left(\frac{\pi t}{6}\right) = \frac{x-10}{3}$, $\sin\left(\frac{\pi t}{6}\right) = y - 5$

$\cos^2\left(\frac{\pi t}{6}\right) + \sin^2\left(\frac{\pi t}{6}\right) = 1$

$\therefore \frac{(x-10)^2}{9} + (y-5)^2 = 1$

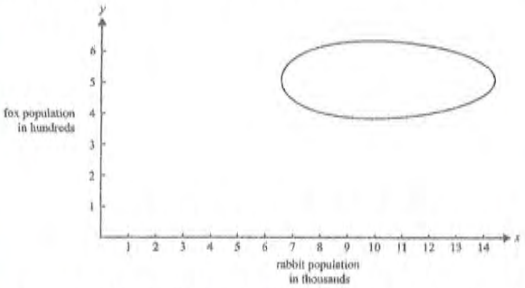
2 marks

b. Sketch the relationship between x and y on the axes below.



2 marks

For certain populations of foxes and rabbits $c = 27.5$. The graph of the solution curve from part d. ii. for this value of c is shown below.



e. Determine the **minimum** and **maximum** numbers of rabbits possible according to the alternative model, correct to the nearest ten rabbits.

$\frac{dx}{dt}$ is undefined

$25x - 5xy = 0 \rightarrow y = 5$ (V) or $x = 0$ (X)

sub $y = 5$ into equation

$25(x - 10) - 25 - x + 10\ln(x) = 27.5$

$\rightarrow x = 6.5871$ or $x = 14.4264$

min: 6590 rabbits
max: 14430 rabbits

3 marks

Total 12 marks

e. i. After how many months from the start of the year is the population of rabbits, x thousand, a minimum?

$x = 10 + 3\cos\left(\frac{\pi t}{6}\right) = 7$

$\rightarrow t = 6$ months

ii. How many foxes are on the island at this time?

when $t = 6$, $y = 5$

$\rightarrow$ 500 foxes

1 + 1 = 2 marks

An alternative model for the interaction of the two populations, which more accurately allows for the dependency of the foxes on the rabbits as a food source, is given by the pair of differential equations

Rabbits $\frac{dx}{dt} = 0.5x - 0.1xy, t \geq 0$

Foxes $\frac{dy}{dt} = -0.2y + 0.02xy, t \geq 0$

d. i. Show that $\frac{dy}{dx} = \frac{xy - 10y}{25x - 5xy}$.

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{0.02xy - 0.2y}{0.5x - 0.1xy}$

$\left(\times \frac{50}{50}\right) = \frac{xy - 10y}{25x - 5xy}$ (Shown)

ii. Use calculus to verify that the curve with equation

$25\log_e(y) - 5y = x + 10\log_e(x) = c$

where c is a constant of integration, satisfies the differential equation given in part d. i.

$\frac{25}{y} \times \frac{dy}{dx} - 5\left(\frac{dy}{dx}\right) - 1 + \frac{10}{x} = 0$

$\frac{dy}{dx}\left(\frac{25}{y} - 5\right) = 1 - \frac{10}{x} \rightarrow \frac{dy}{dx}\left(\frac{25 - 5y}{y}\right) = \frac{x - 10}{x}$

$\frac{dy}{dx} = \frac{y(x-10)}{x(25-5y)} = \frac{xy - 10y}{25x - 5xy}$ (Shown)

1 + 2 = 3 marks

VCAA NH 2017 Exam 2 Q3

Bacteria are spreading over a Petri dish at a rate modelled by the differential equation

$$\frac{dP}{dt} = \frac{P}{2}(1-P),\ 0 < P < 1$$

where P is the **proportion** of the dish covered after t hours.

a. i. Express $\frac{2}{P(1-P)}$ in partial fraction form. (1 mark)

$\frac{2}{P} + \frac{2}{1-P}$

ii. Hence show by integration that $\frac{t-c}{2} = \log_e\left(\frac{P}{1-P}\right)$, where c is a constant of integration. (2 marks)

$\frac{dt}{dP} = \frac{2}{P(1-P)} = \frac{2}{P} + \frac{2}{1-P}$

$t = \int\left(\frac{2}{P} + \frac{2}{1-P}\right)dP$

$= 2\log_e|P| - 2\log_e|1-P| + c$

$t - c = 2\log_e\left|\frac{P}{1-P}\right|$

$\frac{t-c}{2} = \log_e\left|\frac{P}{1-P}\right|$

Since $0 < P < 1$, $\boxed{\frac{t-c}{2} = \log_e\left(\frac{P}{1-P}\right)}$ (shown)

iii. If half of the Petri dish is covered by the bacteria at $t = 0$, express P in terms of t. (2 marks)

When $t = 0$, $P = \frac{1}{2}$. $\rightarrow -\frac{c}{2} = \log_e\left(\frac{0.5}{1-0.5}\right)$

$-\frac{c}{2} = \log_e(1) = 0 \rightarrow c = 0$

$\frac{t}{2} = \log_e\left(\frac{P}{1-P}\right) \Rightarrow \frac{P}{1-P} = e^{t/2}$

$\frac{1-P}{P} = e^{-t/2} \rightarrow \frac{1}{P} - 1 = e^{-t/2}$

$\frac{1}{P} = 1 + e^{-t/2}$

$\boxed{P = \frac{e^{t/2}}{1-e^{t/2}}}$

After one hour, a toxin is added to the Petri dish, which harms the bacteria and reduces their rate of growth. The differential equation that models the rate of growth is now

$$\frac{dP}{dt} = \frac{P}{2}(1-P) - \frac{\sqrt{P}}{20} \text{ for } t \geq 1$$

b. Find the limiting value of P, which is the maximum possible proportion of the Petri dish that can now be covered by the bacteria. Give your answer correct to three decimal places. (2 marks)

When $t \rightarrow \infty$, $\frac{dP}{dt} = 0$.

$\frac{P}{2}(1-P) = \frac{\sqrt{P}}{20}$

$\rightarrow P = 0$ (reject), $P = 0.010$ (reject),

$P = 0.894$ (accept)

$\rightarrow \boxed{P = 0.894}$ (3 dps)

c. The total time, T hours, measured from time $t = 0$, needed for the bacteria to cover 80% of the Petri dish is given by

$$T = \int_q^r \left(\frac{1}{\frac{P}{2}(1-P) - \frac{\sqrt{P}}{20}}\right)dP + s$$

where q, r and $s \in R$.

Find the values of q, r and s, giving the value of q correct to two decimal places. (3 marks)

When $t = 1$, $P = 0.62$ (2 dps)

$\rightarrow \boxed{q = 0.62,\ r = 0.80,\ s = 1.00}$ (2 dps)

d. Given that $P = 0.75$ when $t = 3$, use Euler's method with a step size of 0.5 to estimate the value of P when $t = 3.5$. Give your answer correct to three decimal places. (3 marks)

$t_0 = 3$, $P_0 = 0.75$ $\rightarrow$ When $P = 0.75$, $\frac{dP}{dt} = \frac{3}{32} - \frac{\sqrt{3}}{40}$

When $t_1 = 3.5$, $P_1 = 0.75 + 0.5 \times \left(\frac{3}{32} - \frac{\sqrt{3}}{40}\right)$

$\approx \boxed{0.775}$ (3 dps)