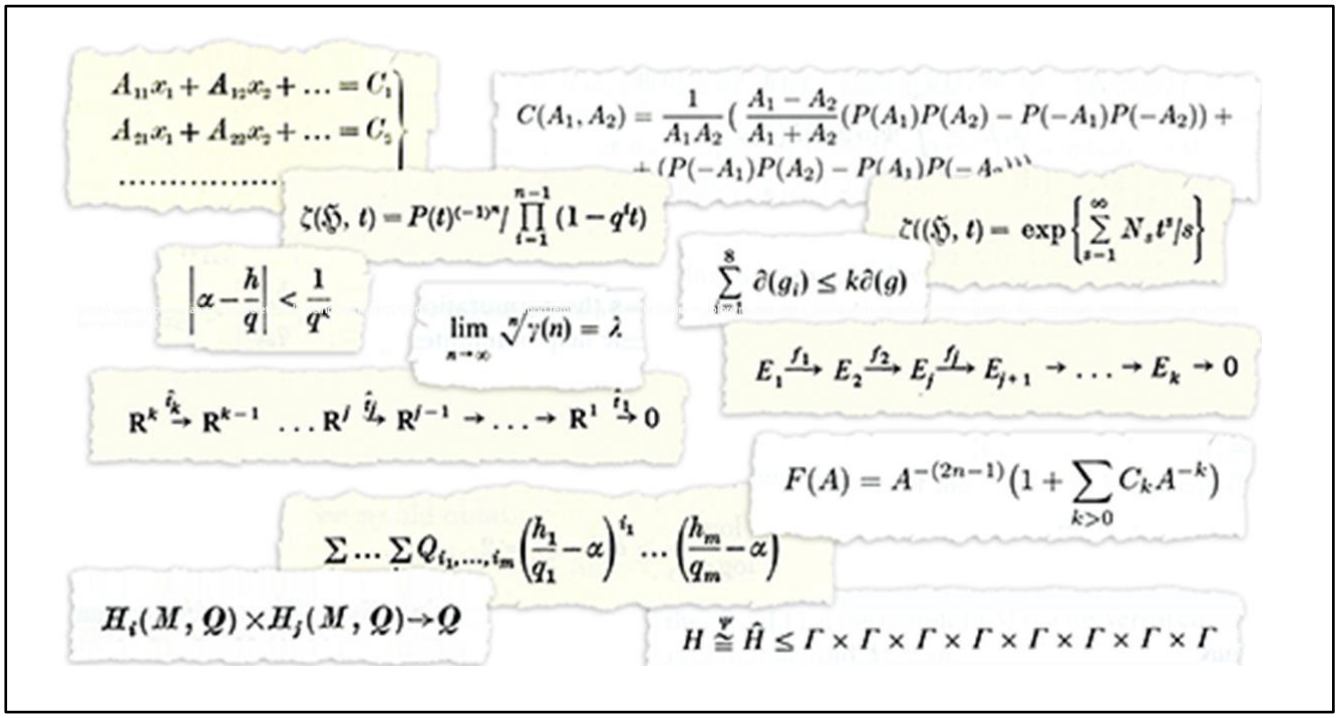
UNIT 3 SPECIALIST MATHS

APPLICATIONS OF INTEGRATION



WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

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Chapter 8 Applications of Integration

Exercise 8A The fundamental theorem of calculus

→ Signed area → 'under' curve (see Ex 7A)

- Definite integral $\int_a^b f(x)\,dx$ (given)
  - regions above x-axis → +ve
  - regions below x-axis → -ve

→ Fundamental theorem of calculus

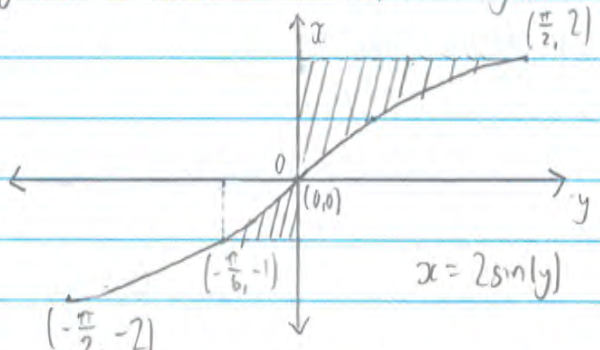
- $\int_c^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$ (Net Change = Accumulated instantaneous rate of change)
  where F is any antiderivative of f

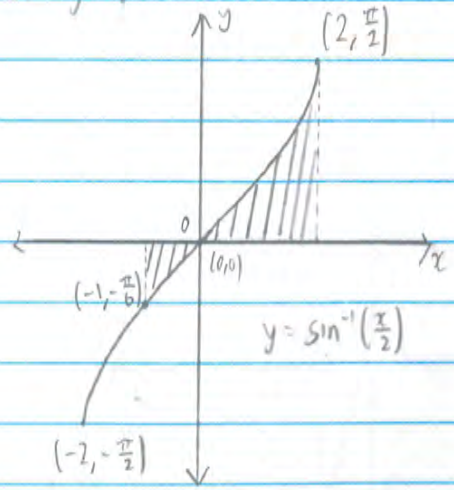
→ Properties of definite integral

- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a < c < b)$
- $\int_a^a f(x)\,dx = 0$
- $\int_a^b h\,f(x)\,dx = h\int_a^b f(x)\,dx$
- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = \int_b^a -f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_{a+h}^{b+h} f(x-h)\,dx$

→ Area under curve examples

- e.g. 1. Evaluate $\int_{-1}^{2} \sin^{-1}\left(\frac{x}{2}\right) dx$ by sketching its graph.





① → $\int_0^2 \sin^{-1}\left(\frac{x}{2}\right) dx = 2 \times \frac{\pi}{2} - \int_0^{\pi/3} 2\sin(y)\,dy$

$= \pi + 2[\cos(y)]_0^{\pi/3} = (\pi - 2)$ units^2

② → $\int_{-1}^{2} \sin^{-1}\left(\frac{x}{2}\right) dx = \int_{-1}^{0} \sin^{-1}\left(\frac{x}{2}\right) dx + \int_0^2 \sin^{-1}\left(\frac{x}{2}\right) dx$

$= (\pi - 2) - \int_0^1 \sin^{-1}\left(\frac{x}{2}\right) dx$

$= (\pi - 2) - \left[\frac{\pi}{6} - \int_0^{\pi/6} 2\sin(y)\,dy\right]$

$= (\pi - 2) - \left[\frac{\pi}{6} + 2[\cos(y)]_0^{\pi/6}\right]$

$= \pi - 2 - \left(\frac{\pi}{6} + \sqrt{3} - 2\right) = \boxed{\left(\frac{5\pi}{6} - \sqrt{3}\right) \text{ units}^2}$

·e.g. 2. Find the area of the region enclosed by the curve $y = \frac{3}{(2x+1)(1-x)}$ and the line $y = 3$.

$\rightarrow \frac{3}{(2x+1)(-x+1)} = \frac{a}{2x+1} + \frac{b}{-x+1}$

$3 = a(-x+1) + b(2x+1)$

Sub $x = 1$, $3 = 3b \rightarrow b = 1$

Sub $x = -\frac{1}{2}$, $3 = \frac{3}{2}a \rightarrow a = 2$

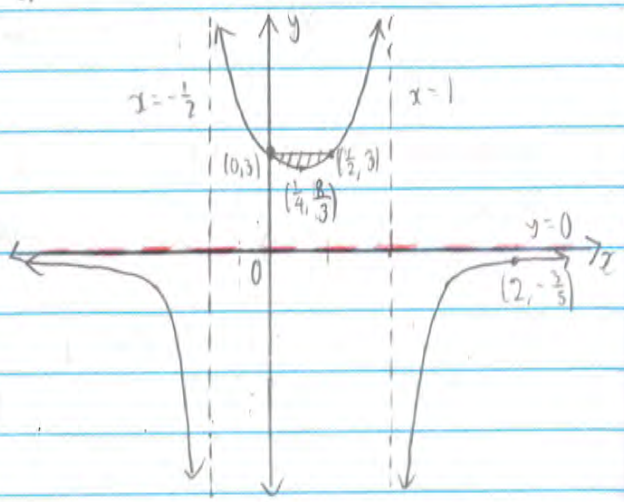
$\rightarrow \frac{2}{2x+1} - \frac{1}{x-1}$

$\rightarrow$ Area $= 3 \times \frac{1}{2} - \int_0^{\frac{1}{2}} \left(\frac{2}{2x+1} - \frac{1}{x-1}\right) dx$

$= \frac{3}{2} - \left[\log_e\left|\frac{2x+1}{x-1}\right|\right]_0^{\frac{1}{2}}$

$= \frac{3}{2} - \log_e(4) + \log_e(1)$

$= \boxed{\left(\frac{3}{2} - 2\log_e(2)\right) \text{ units}^2}$

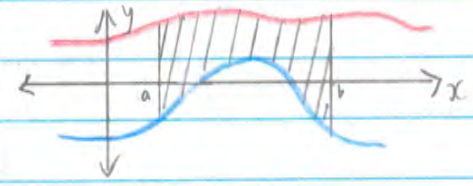


Exercise 8B Area between curves

$\rightarrow$ Area of shaded region (graphs uncross)

· $\int_a^b f(x) - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$

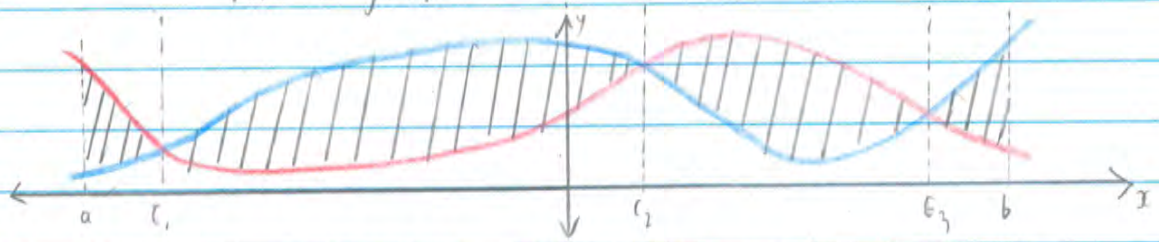
- $f(x) \geq g(x)$ for all $x \in [a, b]$



$\rightarrow$ Area of shaded region (graphs cross)

- Shaded area $= \int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$

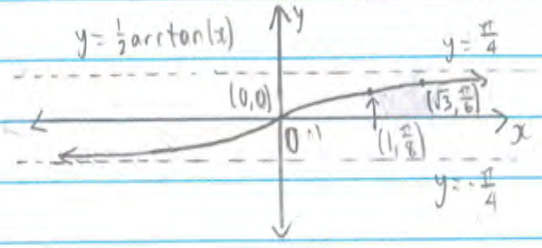
$= \int_a^b |f(x) - g(x)|\,dx$

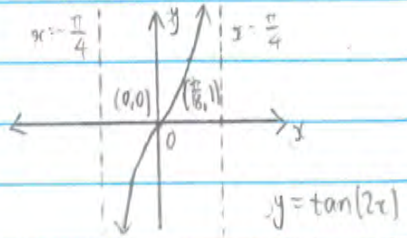


$\rightarrow$ CAS skills: Use modulus function when integrating to find total area

Exercise 8C Integration using a CAS calculator

·e.g. Find the area enclosed by graph of $y = \frac{1}{2}\arctan(x)$, the x-axis and the line $x = \sqrt{3}$ (VCAA 2015 Exam 1 Q8)





→ Area $= \int_0^{\sqrt{3}} \frac{1}{2}\arctan(x)\,dx$

$= \sqrt{3} \times \frac{\pi}{6} - \int_0^{\pi/6} \tan(2x)\,dx$ (inverse function used)

$= \frac{\sqrt{3}\pi}{6} - \left[\frac{1}{2}\log_e|\sec(2x)|\right]_0^{\pi/6}$ (since $\int \tan(2x)\,dx = \frac{1}{2}\log_e|\sec(2x)| + c$)

$= \boxed{\frac{\sqrt{3}\pi}{6} - \frac{1}{2}\log_e(2)}$ (see Ex 7C e.g. 2)

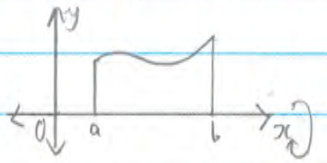
Exercise 8D Volumes of solids of revolution

→ Solids of revolution (solids formed by rotating a region about a line)

→ Volumes of revolution (N.B. Don't forget the π!)

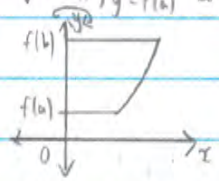
• Rotation about x-axis

$V = \pi\int_a^b y^2\,dx = \pi\int_a^b [f(x)]^2\,dx$



• Rotation about y-axis

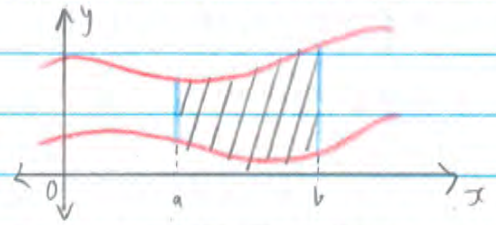
$V = \pi\int_{y=f(a)}^{y=f(b)} x^2\,dy = \left|\pi\int_a^b x^2\left(\frac{dy}{dx}\right)dx\right|$



• Regions not bounded by x-axis

$V = \pi\int_a^b [f(x)]^2 - [g(x)]^2\,dx$

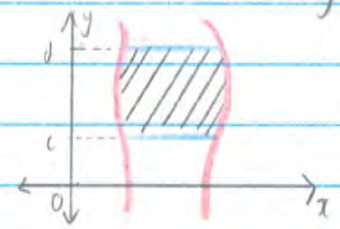
$= \pi\int_a^b (y_2)^2 - (y_1)^2\,dx$



• Regions not bounded by y-axis

$V = \pi\int_c^d [f(y)]^2 - [g(y)]^2\,dy$

$= \pi\int_c^d (x_2)^2 - (x_1)^2\,dy$



* NOT $V = \pi\int_a^b (f(x) - g(x))^2\,dx$!

→ Complex examples (N.B. Always draw a sketch!)

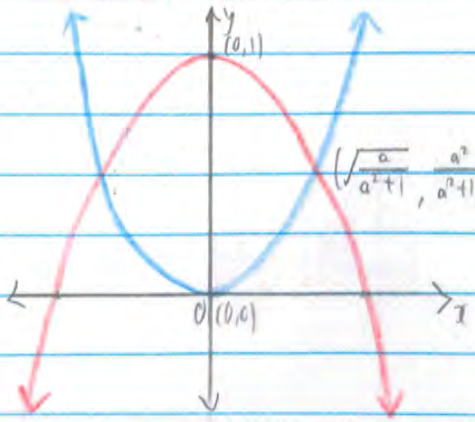
e.g. 1. The curves $y = ax^2$ and $y = 1 - \frac{x^2}{a}$ are shown, where $a > 0$.

(a) Show that the area enclosed by the two curves is $\frac{4}{3}\sqrt{\frac{a}{a^2+1}}$.

(b) (i) Find the value of a which gives the maximum area.

(ii) Find the maximum area

(c) Find the volume of the solid formed when the region bounded by these curves is rotated around the y-axis



(a) $\text{Area} = \int_{-\sqrt{\frac{a}{a^2+1}}}^{\sqrt{\frac{a}{a^2+1}}} \left(1 - \frac{x^2}{a} - ax^2\right) dx = 2\int_0^{\sqrt{\frac{a}{a^2+1}}} \left(1 - \frac{x^2}{a} - ax^2\right) dx$

$= 2\left[x - \frac{x^3}{3a} - \frac{1}{3}ax^3\right]_0^{\sqrt{\frac{a}{a^2+1}}}$

$= \frac{2\sqrt{\frac{a}{a^2+1}}}{3}\left(3 - \frac{a}{a(a^2+1)} - a\left(\frac{a}{a^2+1}\right)\right)$

$= \frac{2\sqrt{\frac{a}{a^2+1}}}{3}\left(3 - \frac{1+a^2}{a^2+1}\right) = \frac{2\sqrt{\frac{a}{a^2+1}}}{3}(3-1)$

$= \boxed{\frac{4}{3}\sqrt{\frac{a}{a^2+1}}}$ (Shown)

(b) (i) $A = \frac{4}{3}\sqrt{\frac{a}{a^2+1}} \rightarrow \frac{dA}{da} = \frac{2(1-a^2)}{3\sqrt{a}(a^2+1)^{3/2}}$

$\frac{dA}{da} = 0$ when $\boxed{a = 1}$ (since $a > 0$)

(ii) When $a = 1$, $A = \boxed{\frac{2\sqrt{2}}{3} \text{ units}^2}$

(c) For $y = ax^2$, $x^2 = \frac{y}{a}$.

For $y = 1 - \frac{x^2}{a}$, $x^2 = a(1-y)$

$V = \pi\int_0^{\frac{a^2}{a^2+1}} \left(\frac{y}{a}\right) dy + \pi\int_{\frac{a^2}{a^2+1}}^1 (a(1-y))\, dy$

$= \frac{\pi}{a}\left[\frac{1}{2}y^2\right]_0^{\frac{a^2}{a^2+1}} + a\pi\left[y - \frac{1}{2}y^2\right]_{\frac{a^2}{a^2+1}}^1$

$= \frac{\pi a^3}{2(a^2+1)^2} + \frac{\pi a}{2(a^2+1)^2} = \frac{\pi a(a^2+1)}{2(a^2+1)^2}$

$= \boxed{\frac{\pi a}{2(a^2+1)} \text{ units}^3}$

e.g. 2. Find the volume of the solid formed when the region bounded by the x-axis and the curve with equation $y = a - \frac{x^2}{16a^3}$, where $a > 0$, is rotated around the y-axis.

→ Rearranging equation, $x^2 = 16a^3(a-y) = 16a^4 - 16a^3y$

→ x-intercepts ($y = 0$): $0 = a - \frac{x^2}{16a^3} \rightarrow x^2 = 16a^4 \rightarrow x = \pm 4a^2$

When $x = 0$, $y = a$. When $x = 4a^2$, $y = 0$.

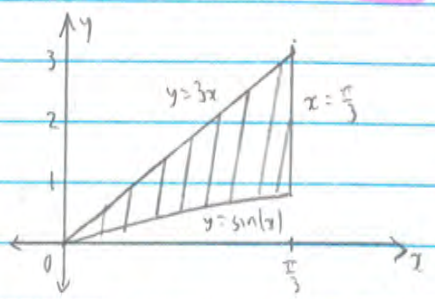
→ $\text{Volume} = \pi\int_0^{a^2} x^2\, dy = \pi\int_0^a (16a^4 - 16a^3y)\, dy$

$= 16a^3\pi\int_0^a (a-y)\, dy$

$= 16a^3\pi\left[ay - \frac{1}{2}y^2\right]_0^a$

$= 16a^3\pi\left(\frac{1}{2}a^2\right) = \boxed{8\pi a^5 \text{ units}^3}$

· e.g. 3. The shaded region below is enclosed by the graph of $y = \sin(x)$ and the lines $y = 3x$ and $x = \frac{\pi}{3}$. The region is rotated about x-axis. Find the volume of resulting solid of revolution. (VCAA 2013 Exam 1 Q9)



$\rightarrow V = \pi \int_0^{\pi/3} (3x)^2 \, dx - \pi \int_0^{\pi/3} (\sin^2(x)) \, dx$

$= \pi \int_0^{\pi/3} (9x^2) \, dx - \pi \int_0^{\pi/3} \frac{1}{2}(1 - \cos(2x)) \, dx$

$= \pi \left[3x^3\right]_0^{\pi/3} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi/3}$

$= 3\pi\left(\frac{\pi}{3}\right)^3 - \frac{\pi}{2}\left(\frac{\pi}{3} - \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)\right)$

$= 3\pi\left(\frac{\pi^3}{27}\right) - \frac{\pi}{6} + \frac{\pi}{4}\left(\frac{\sqrt{3}}{2}\right)$

$\therefore = \boxed{\left(\frac{1}{9}\pi^4 + \frac{\sqrt{3}}{8}\pi^2 - \frac{1}{6}\pi\right) \text{units}^3}$

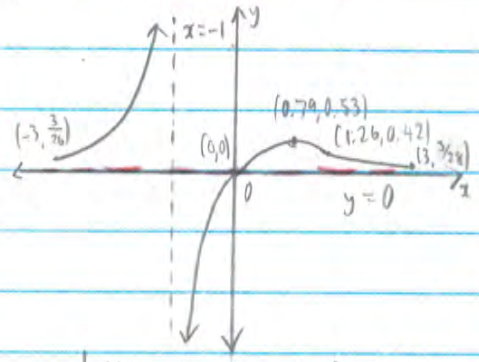
· e.g. 4. Let $f: R\setminus\{-1\} \rightarrow R$, $f(x) = \frac{x}{1+x^3}$ (VCAA 2017 Exam 2 Q1)

The region S, bounded by the graph of f, the x-axis to form a solid of revolution.

The line $x = a$, where $0 < a < 3$, divides the region S into two regions such that, when the two regions are rotated about the x-axis, they generate solids of equal volume.

Write down an equation involving definite integrals that can be used to determine a.

Hence, find the value of a, correct to two decimal places.



Key Features: (2 dps)
Asymptotes: $x = -1$, $y = 0$
Stationary points: (0.79, 0.53)
Point of inflection: (1.26, 0.42) (not (0,0)!)

$\rightarrow \pi \int_0^a \left(\frac{x}{1+x^3}\right)^2 dx = \pi \int_a^3 \left(\frac{x}{1+x^3}\right)^2 dx$

$\rightarrow$ Solve on CAS, $\boxed{a \approx 0.98}$ (2 dps)

· e.g. 5. VCAA 2017 Exam 1 Q10

(a) Show that $\frac{d}{dx}\left(x \arccos\left(\frac{x}{a}\right)\right) = \arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}$, where $a > 0$.

$\rightarrow$ Let $u = x$ and $v = \arccos\left(\frac{x}{a}\right)$

$\frac{du}{dx} = 1$ $\quad$ $\frac{dv}{dx} = \frac{-1}{\sqrt{a^2 - x^2}}$

$\rightarrow \frac{d}{dx}\left(x \arccos\left(\frac{x}{a}\right)\right) = \arccos\left(\frac{x}{a}\right) + x \times \frac{-1}{\sqrt{a^2 - x^2}} = \boxed{\arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}}$ (Shown)

(b) State the maximal domain and range of $f(x) = \sqrt{\arccos\left(\frac{x}{2}\right)}$

$\rightarrow$ Domain: $\boxed{x \in [-2, 2]}$

$\rightarrow$ Range: $\boxed{y \in [0, \sqrt{\pi}]}$

(c) Find the volume of the solid of revolution generated when the region bounded by the graph of $y = f(x)$, and the lines $x = -2$ and $y = 0$, is rotated about the x-axis.

→ From (a), $\int \left(\arccos\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2 - x^2}}\right) dx = x\arccos\left(\frac{x}{a}\right) + c$

$$\int \arccos\left(\frac{x}{a}\right) dx = x\arccos\left(\frac{x}{a}\right) + \int \frac{x}{\sqrt{a^2 - x^2}}\, dx + c$$

$$= x\arccos\left(\frac{x}{a}\right) - \sqrt{a^2 - x^2} + c$$

$$\left[\text{since } \frac{d}{dx}\left(-(a^2 - x^2)^{1/2}\right) = -\frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{a^2 - x^2}}\right]$$

→ Volume $= \pi \int_{-2}^{2} y^2\, dx = \pi \int_{-2}^{2} \arccos\left(\frac{x}{2}\right) dx$

$$= \pi\left[x\arccos\left(\frac{x}{2}\right) - \sqrt{4 - x^2}\right]_{-2}^{2}$$

$$= \pi\left[2\arccos(1) - (-2)\arccos(-1)\right]$$

$$= \pi\left(2(0) + 2(\pi)\right) = \boxed{2\pi^2 \text{ units}^3}$$

Exercise 8E Length of curves in the plane

→ Length of curve: $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

→ Length of parametric curve $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

→ CAS skills: arcLen(expression, variable, start, end)

from Menu → Calculus(4) → Arc Length(8)

→ Length of curve examples

e.g. 1. Given that $f(x) = \log_e(\sec(x) + \tan(x))$, find $f'(x)$. Hence find the length of the curve $y = \log_e(\cos(x))$ for $0 \le x \le \frac{\pi}{4}$.

→ Let $w = \log_e(\sec(x) + \tan(x))$. Let $u = \sec(x) + \tan(x)$ so that $w = \log_e(u)$.

$$\frac{du}{dx} = \frac{\sin(x)}{\cos^2(x)} + \sec^2(x) = \sec(x)\tan(x) + \sec^2(x), \quad \frac{dw}{du} = \frac{1}{u}$$

$$\frac{dw}{dx} = \frac{\sec(x)(\tan(x) + \sec(x))}{\tan(x) + \sec(x)} = \sec(x) \rightarrow \boxed{f'(x) = \sec(x)}$$

→ $y = \log_e(\cos(x)) \rightarrow \frac{dy}{dx} = \frac{-\sin(x)}{\cos(x)} = -\tan(x)$

→ $L = \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^{\pi/4} \sqrt{1 + \tan^2(x)}\, dx$

$$= \int_0^{\pi/4} \sqrt{\sec^2(x)}\, dx = \int_0^{\pi/4} \sec(x)\, dx \quad \left(\text{since } 0 \le x \le \tfrac{\pi}{4}\right)$$

$$= \left[\log_e(\sec(x) + \tan(x))\right]_0^{\pi/4} = \boxed{\log_e(\sqrt{2} + 1)}$$

- e.g. 2. Find the length of the curve specified parametrically by $x = a\theta - a\sin(\theta)$, $y = a - a\cos(\theta)$ from $\theta = \frac{2\pi}{3}$ to $\theta = 2\pi$, where $a \in R^+$. Give your answer in terms of a. (VCAA NH 2017 Exam 1 Q11)

$\rightarrow \frac{dx}{d\theta} = a - a\cos(\theta) \quad \frac{dy}{d\theta} = a\sin(\theta)$

$\rightarrow L = \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{(a - a\cos\theta)^2 + (a\sin\theta)^2}\, d\theta$

$= \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{a^2 - 2a^2\cos(\theta) + a^2\cos^2(\theta) + a^2\sin^2\theta}\, d\theta$

$= \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{2a^2 - 2a^2\cos(\theta)}\, d\theta$

$= \sqrt{2}\, a \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{1 - \cos(\theta)}\, d\theta$

$= \sqrt{2}\, a \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{1 - \left(1 - 2\sin^2\left(\frac{\theta}{2}\right)\right)}\, d\theta$ (double angle formula: $\cos(2x) = 1 - 2\sin^2(x)$)

$= 2a \int_{\frac{2\pi}{3}}^{2\pi} \sqrt{\sin^2\left(\frac{\theta}{2}\right)}\, d\theta$

$= 2a \int_{\frac{2\pi}{3}}^{2\pi} \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$

$= -4a\left[\cos\left(\frac{\theta}{2}\right)\right]_{\frac{2\pi}{3}}^{2\pi} = -4a\left(\cos(\pi) - \cos\left(\frac{\pi}{3}\right)\right)$

$= -4a\left(-1 - \frac{1}{2}\right) = \boxed{6a}$

- e.g. 3. The position of a particle at time t seconds is given by (see Ex 12D) $\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + \frac{4t^2}{5}\sqrt{t}\,\underset{\sim}{j}$, $t \geq 0$, where components are measured in metres. Find the distance travelled in the first 3 seconds of its motion.

$\rightarrow$ Arc length $= \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$

$\rightarrow D = \int_0^3 \sqrt{(2t)^2 + (2t^{3/2})^2}\, dt \quad \left(\underset{\sim}{r}(t) = t^2\underset{\sim}{i} + \frac{4}{5}t^{5/2}\underset{\sim}{j}\right)$

$= \int_0^3 \sqrt{4t^2 + 4t^3}\, dt \quad \left(\underset{\sim}{v}(t) = 2t\underset{\sim}{i} + 2t^{3/2}\underset{\sim}{j}\right)$

$= \int_0^3 \sqrt{4t^2(1+t)}\, dt$

$= \int_0^3 2t\sqrt{1+t}\, dt \quad (\text{since } \sqrt{4t^2} = |2t| = 2t,\ t \geq 0)$

Let $u = t + 1 \rightarrow \frac{du}{dt} = 1 \rightarrow du = dt$

When $t = 0$, $u = 1$. When $t = 3$, $u = 4$

$D = 2\int_1^4 (u-1)\sqrt{u}\, du = 2\int_1^4 \left(u^{3/2} - u^{1/2}\right) du$

$= 2\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^4$

$= 2\left(\frac{2}{5}(32) - \frac{2}{3}(8) - \frac{2}{5} + \frac{2}{3}\right)$

$= 2\left(\frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3}\right) = 2\left(\frac{62}{5} - \frac{14}{3}\right)$

$= 2\left(\frac{186 - 70}{15}\right)$

$D = \boxed{\frac{232}{15} \text{ metres}}$

Challenge of Volumes of Revolution (2.0)

The region enclosed by the graph of $y = \frac{2}{\sqrt{x}} \cdot \frac{1}{\sqrt{3+\sqrt{x}}}$ and the lines $y = 0$, $x = 1$ and $x = 4$ is rotated about the x-axis to form a solid revolution of volume V. By making an appropriate substitution in the above integral calculate the value of V. Give your answer in the form $2a\left(1 + b\ln\left(\frac{a}{5}\right)\right)$, where a and b are positive integers.

$\rightarrow V = \pi\int_1^4 y^2 \, dx = \pi\int_1^4 \frac{4}{\pi}\left(\frac{1}{3+\sqrt{x}}\right) dx$

$= 4\int_1^4 \frac{1}{3+\sqrt{x}} \, dx$

Method 1: Let $u = \sqrt{x} \rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2u} \rightarrow dx = 2u\,du$

$x = 1 \rightarrow u = 1$

$x = 4 \rightarrow u = \sqrt{4} = 2$

$V = 4\int_1^2 \frac{2u}{3+u} \, du = 8\int_1^2 \frac{u}{u+3} \, du$

$= 8\int_1^2 \left(1 - \frac{3}{u+3}\right) du$

$= 8\left[u - 3\log_e|u+3|\right]_1^2$

$= 8\left[\left[2 - 3\log_e(5)\right] - \left[1 - 3\log_e|4|\right]\right]$

$= \boxed{8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right) \text{ units}^3}$ $(a = 4, b = 3)$

Method 2: Let $u = 3 + \sqrt{x} \rightarrow \frac{du}{dx} = \frac{1}{2\sqrt{x}} = \frac{1}{2(u-3)} \rightarrow dx = 2(u-3)\,du$

$x = 1 \rightarrow u = 3 + \sqrt{1} = 4$

$x = 4 \rightarrow u = 3 + \sqrt{4} = 5$

$V = 4\int_4^5 \frac{2(u-3)}{u} \, du = 8\int_4^5 \frac{u-3}{u} \, du$

$= 8\int_4^5 \left(1 - \frac{3}{u}\right) du$

$= 8\left[u - 3\log_e|u|\right]_4^5$

$= 8\left[\left[5 - 3\log_e(5)\right] - \left[4 - 3\log_e|4|\right]\right]$

$= \boxed{8\left(1 + 3\log_e\left(\frac{4}{5}\right)\right) \text{ units}^3}$ $(a = 4, b = 3)$

→ Chapter 9 Complex Example (Ex 9D)

In a chemical reaction, a compound C (kg) is formed by combining one part of chemical L and two parts of chemical M. The rate at which C is formed is proportional to the product of the amounts of chemicals L and M that are still present in the mixture at any given time t minutes. Initially, there are 5 kg of L, 8 kg of M and no amount of C present, but after some time, x kg of compound C has been formed.

a. How much of this x kg of C formed is chemical M and how much is chemical L?

1 mark

$M = \frac{2x}{3}$ kg

$L = \frac{x}{3}$ kg

b. How many kilograms of chemical M and chemical L remain in the mixture?

1 mark

$M = \left(8 - \frac{2x}{3}\right)$ kg

$L = \left(5 - \frac{x}{3}\right)$ kg

c. Show that $\frac{dx}{dt} = k(12-x)(15-x)$, where k is a constant.

2 marks

$\frac{dx}{dt} \propto \left(8 - \frac{2x}{3}\right)\left(5 - \frac{x}{3}\right)$

$\frac{dx}{dt} \propto \frac{1}{9}(24x - 2x)(15 - x)$

$\frac{dx}{dt} \propto \frac{2}{9}(12 - x)(15 - x)$

$\frac{dx}{dt} = k(12 - x)(15 - x)$ (shown)

(d) After 1 min, 2 kg of chemical C formed. Using calculus, solve the differential equation, writing t in terms of x (7 marks)

Question 3d.

Worked solution

$\frac{dx}{dt} = k(12-x)(15-x)$

$\frac{dt}{dx} = \frac{1}{k(12-x)(15-x)}$ (1m)

$t = \frac{1}{k}\int \frac{1}{(12-x)(15-x)}\,dx$

Now $\frac{1}{(12-x)(15-x)} = \frac{A}{12-x} + \frac{B}{15-x}$ (1m)

$= \frac{A(15-x) + B(12-x)}{(12-x)(15-x)}$

$= \frac{15A - Ax + 12B - Bx}{(12-x)(15-x)}$

$\Rightarrow -A - B = 0 \quad (1)$

$15A + 12B = 1 \quad (2)$ (1m)

Solving equations (1) and (2) gives $A = \frac{1}{3}$ and $B = \frac{-1}{3}$.

$t = \frac{1}{k}\int \frac{\frac{1}{3}}{12-x} - \frac{\frac{1}{3}}{15-x}\,dx$

$t = \frac{1}{3k}\int \frac{1}{12-x} - \frac{1}{15-x}\,dx$

$t = \frac{1}{3k}[\log_e|15-x| - \log_e|12-x|] + c, \quad 0 \le x < 12$ (1A)

$t = \frac{1}{3k}\log_e\left|\frac{15-x}{12-x}\right| + c, \quad 0 \le x < 12$

Now $x = 0$ when $t = 0$

$0 = \frac{1}{3k}\log_e\left|\frac{5}{4}\right| + c$

$\Rightarrow c = \frac{-1}{3k}\log_e\left|\frac{5}{4}\right|$ (1A)

$t = \frac{1}{3k}\log_e\left|\frac{15-x}{12-x}\right| - \frac{1}{3k}\log_e\left|\frac{5}{4}\right|$

$t = \frac{1}{3k}\log_e\left|\frac{60-4x}{60-5x}\right|$

Now $x = 2$ when $t = 1$

$1 = \frac{1}{3k}\log_e\left|\frac{26}{25}\right|$

$k = \frac{1}{3}\log_e\left|\frac{26}{25}\right|$ (1A)

$\therefore t = \frac{1}{\log_e\left|\frac{26}{25}\right|}\log_e\left|\frac{60-4x}{60-5x}\right|$ (1A)

VCAA 2003 Exam 2 Extension Question 5 (14 marks) (Chapter 9)

Nick takes a bottle of milk from the refrigerator for baby Alex. To heat the bottle, Nick puts it in a saucepan of continuously boiling water. Let y °C be the temperature of the milk at time t minutes after the baby's bottle is placed in the boiling water.

A differential equation that models the increase in temperature of the milk while the bottle is in the boiling water is $\frac{dy}{dx} = a(100 - y), where\ a > 0$

a) The milk's temperature when the bottle is put into the boiling water is 5°C.
Solve the differential equation to show that $y = 100 - 95e^{-at}, for\ 0 \le t \le T$, where T is the time when Nick takes the bottle out of the boiling water. (4 marks

$\frac{dy}{dt} = a(100 - y)$

$\frac{dt}{dy} = -1\frac{-1}{a(100-y)}$

$t + c = -\frac{1}{a}\log_e|100 - y|$

let $-ca = d$

$-at + d = \log_e|100 - y|$

$e^{-at+d} = |100 - y|$

$e^d = A$

$Ae^{-at} = 100 - y$

$y = 100 - Ae^{-at}$

We know that at $t = 0, y = 5$

$5 = 100 - A$

$\Rightarrow A = 95$

$\therefore y = 100 - 95e^{-at}$ (as required)

When Nick takes the bottle out of the boiling water at time T, the temperature of the milk is 48°C. He realises that this is too hot to give to baby Alex and so he puts the bottle into cold water. The temperature of the cold water is 10°C and the milk cools according to Newton's law of cooling: $\frac{dy}{dt} = -b(y - 10), where\ b > 0$

b) Verify, by differentiation, that for $t \ge T$, $y = 10 + Ae^{-b(t-T)}$, and evaluate A (3 marks)

b $y = 10 + Ae^{-b(t-T)}$

$\frac{dy}{dt} = -bAe^{-b(t-T)}$

From 1 we know that $Ae^{-b(t-T)} = y - 10$

$\therefore \frac{dy}{dt} = -b(y - 10)$

To evaluate A, use the initial conditions, that is: at $t = T, y = 48$

$y = 10 + Ae^{-b(t-T)}$

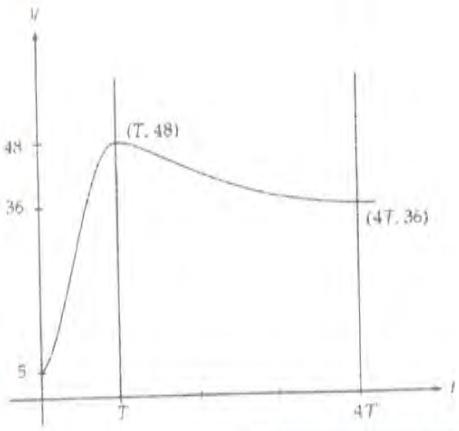
$\therefore 48 = 10 + Ae^0$

$A = 38$

$\therefore y = 10 + 38Ae^{-b(t-T)}$

Nick lets the milk cool to a temperature of 36°C to give to baby Alex. It takes three times as long for the milk to cool to this temperature from 48°C as it previously took to heat up from 5°C to 48°C.

c) Sketch a graph of y in terms of t from when the baby's bottle is put into the boiling water to when the milk is ready to give to baby Alex. (3 marks)



d. Find the ratio $\frac{a}{b}$ correct to 3 sig figs (4 marks)

d $y = 100 - 95e^{-at}$

at $t = T, y = 48$

$48 = 100 - 95e^{-aT}$

$-52 = -95e^{-aT}$

$\frac{52}{95} = e^{-aT}$

$-aT = \log_e\left(\frac{52}{95}\right)$ ... (1)

$y = 10 + 38e^{-b(t-T)}$

at $t = 4T, y = 36$

$36 = 10 + 38e^{-3bT}$

$26 = 38e^{-3bT}$

$-3bT = \log_e\left(\frac{26}{38}\right)$ ... (2)

$1 \div 2$

$\frac{aT}{3bT} = \frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$

$\frac{a}{b} = 3\frac{\log_e\left(\frac{52}{95}\right)}{\log_e\left(\frac{26}{38}\right)}$

$= 4.76$ (correct to 3 significant figures)

VCAA 2011 Exam 2 Question 16 (25% received 1 mark)

The gradient of the normal to a curve at any point $P(x, y)$ is twice the gradient of the line joining P and the point Q(1,1). The coordinates of points on the curve satisfy the differential equation:

A. $\frac{dy}{dx} + \frac{x-1}{2(y-1)} = 0$

B. $\frac{dy}{dx} - \frac{x-1}{2(y-1)} = 0$

C. $\frac{dy}{dx} + \frac{2(y-1)}{x-1} = 0$

D. $\frac{dy}{dx} + \frac{2(x-1)}{y-1} = 0$

E. $\frac{dy}{dx} - \frac{2(y-1)}{x-1} = 0$

Gradient of tangent $= \frac{dy}{dx}$

Gradient of the normal $-\frac{1}{\frac{dy}{dx}} = 2 \times \frac{y-1}{x-1}$

$\therefore \frac{dy}{dx} = -\frac{x-1}{2(y-1)}\ thus\ \frac{dy}{dx} + \frac{x-1}{2(y-1)} = 0$