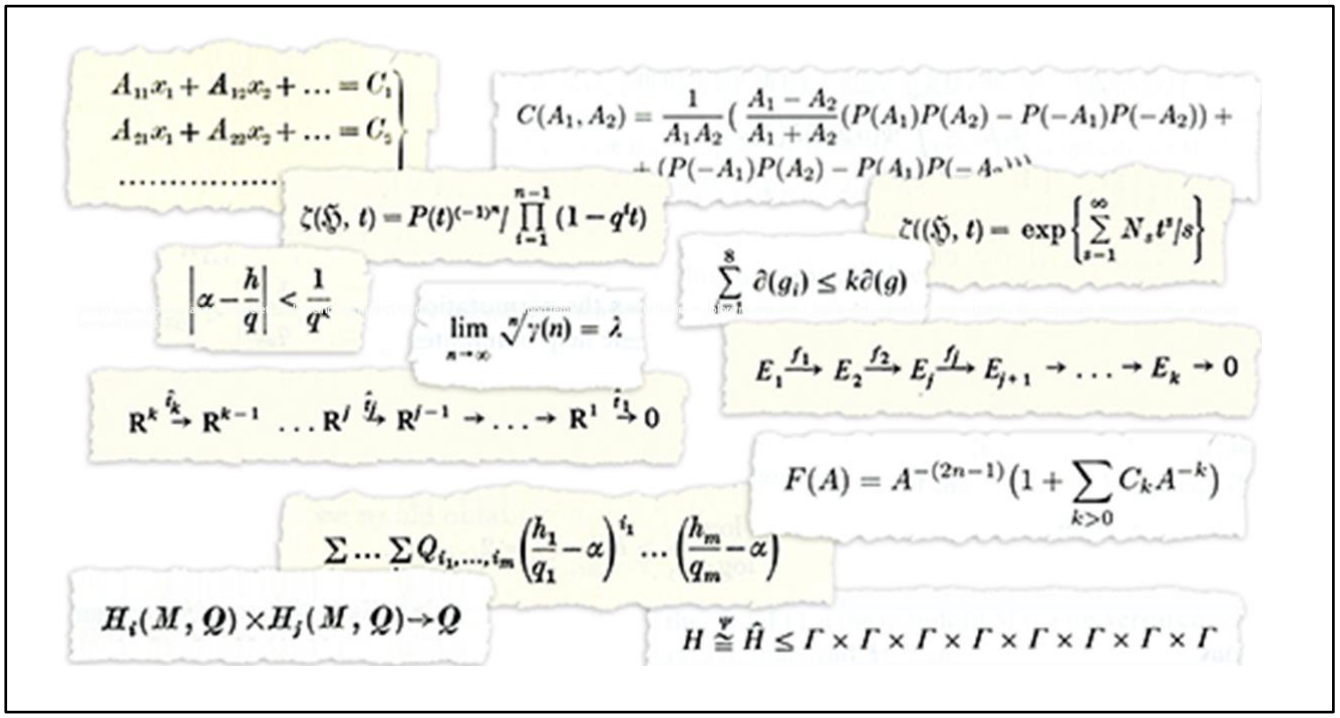
UNIT 3 SPECIALIST MATHS

TECHNIQUES IN INTEGRATION



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Chapter 7 Techiques of Integration

Exercise 7A Antidifferentiation

→ Definition: process of finding a function from its derivative

- $\int f(x)\,dx = F(x) + c$, if $F'(x) = f(x)$

→ Definite integral: $\int_a^b f(x)\,dx = F(b) - F(a)$

- signed area enclosed by graph of $y = f(x)$, x-axis and lines $x = a$ and $x = b$
- integrand → function, lower limit → a, upper limit → b

→ Antiderivatives (also from Mathematical Methods Units 3 and 4)

$f(x)$	$\int f(x)\,dx$, $F(x)$	
x^n	$\frac{1}{n+1}x^{n+1} + c$	, $n \neq -1$
$(ax+b)^n$	$\frac{1}{a(n+1)}(ax+b)^{n+1} + c$	, $n \neq -1$
$\frac{1}{ax+b}$	$\frac{1}{a}\log_e\lvert ax+b\rvert + c$	, $ax+b \neq 0$
e^{ax+b}	$\frac{1}{a}e^{ax+b} + c$	
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b) + c$	
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b) + c$	
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b) + c$	
$\operatorname{cosec}^2(ax+b)$	$-\frac{1}{a}\cot(ax+b) + c$	
$\cot(ax+b)$	$\frac{1}{a}\log_e\lvert\sin(ax+b)\rvert + c$	
$\tan(ax+b)$	$-\frac{1}{a}\log_e\lvert\cos(ax+b)\rvert + c$	

- e.g. 1. Find an antiderivative of $\frac{2x+1}{x+3}$.

→ $\int\left(\frac{2x+1}{x+3}\right)dx = \int\left(2 - \frac{5}{x+3}\right)dx$

$\therefore = \boxed{2x - 5\log_e\lvert x+3\rvert + c} = \boxed{2x - 5\log_e\lvert x+3\rvert}$ (Let $c = 0$)

Steps for an antiderivative:
① Integrate
② Write with +c
③ Let c = 0 → Final ANS

- e.g. 2. Evaluate $\int_{-1}^{0}\left(\frac{1}{4-3x}\right)dx$.

→ $\left[-\frac{1}{3}\log_e\lvert 4-3x\rvert\right]_{-1}^{0} = -\frac{1}{3}\left(\log_e(4) - \log_e(7)\right) = \boxed{\frac{1}{3}\log_e\left(\frac{7}{4}\right)}$

Exercise 7B Antiderivatives involving inverse circular functions

→ Rules:

$\int \frac{a}{\sqrt{b^2 - c^2x^2}}\,dx = \frac{a}{c}\sin^{-1}\left(\frac{c}{b}x\right) + C = -\frac{a}{c}\cos^{-1}\left(\frac{c}{b}\right)x + C$, $x \in \left(-\frac{b}{c}, \frac{b}{c}\right)$

$\int \frac{a}{b^2 + c^2x^2}\,dx = \frac{a}{bc}\tan^{-1}\left(\frac{c}{b}x\right) + C$, $x \in R$

$\int \frac{1}{\sqrt{a^2 - (bx+c)^2}}\,dx = \frac{1}{b}\sin^{-1}\left(\frac{bx+c}{a}\right) + C = -\frac{1}{b}\cos^{-1}\left(\frac{bx+c}{a}\right) + C$, $x \in \left(\frac{-a-c}{b}, \frac{a-c}{b}\right)$

$\int \frac{1}{a^2 + (bx+c)^2}\,dx = \frac{1}{ab}\tan^{-1}\left(\frac{bx+c}{a}\right) + C$, $x \in R$

→ Antiderivative examples

- e.g. 1. $\int\left(\frac{1}{\sqrt{5-2x^2}}\right)dx = \int\left(\frac{1}{\sqrt{2\left(\frac{5}{2}-x^2\right)}}\right)dx = \frac{\sqrt{2}}{2}\int\left(\frac{1}{\sqrt{\frac{5}{2}-x^2}}\right)dx =$

$\therefore = \frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{x}{\sqrt{\frac{5}{2}}}\right)+C = \boxed{\frac{\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{10}}{5}x\right)+C}$

- e.g. 2. $\int\left(\frac{7}{3+y^2}\right)dy = \frac{7}{\sqrt{3}}\int\left(\frac{\sqrt{3}}{3+y^2}\right)dy$

$\therefore = \boxed{\frac{7\sqrt{3}}{3}\tan^{-1}\left(\frac{\sqrt{3}}{3}y\right)+C}$

- e.g. 3. $\int\left(\frac{-1}{\sqrt{1-4x^2}}\right)dx = -\frac{1}{2}\int\left(\frac{2}{\sqrt{1-(2x)^2}}\right)dx = \boxed{\frac{1}{2}\cos^{-1}(2x)+C}$

Exercise 7C Integration by substitution (change of variable rule)

→ Linear substitution: $\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$

- e.g. 1. Find an antiderivative of $\frac{\cos(x)}{\sin^5(x)}$.

Let $u = \sin(x) \rightarrow \frac{du}{dx} = \cos(x) \rightarrow du = \cos(x)\,dx$

$\int\frac{\cos(x)}{\sin^5(x)}dx = \int\frac{1}{u^5}du = -\frac{1}{4u^4}+C$

$= -\frac{1}{4\sin^4(x)}+C = \boxed{-\frac{1}{4}\operatorname{cosec}^4(x)+C} = \boxed{-\frac{1}{4}\operatorname{cosec}^4(x)}$ (Let $C=0$)

- e.g. 2. Find $\int(\tan(2x))dx$ (VCAA 2015 Exam 1 Q 8)

Let $u = \cos(2x) \rightarrow \frac{du}{dx} = -2\sin(2x) \rightarrow du = (-2\sin(2x))\,dx$

$\rightarrow \int(\tan(2x))dx = \int\left(\frac{\sin(2x)}{\cos(2x)}\right)dx = -\frac{1}{2}\int\left(\frac{1}{u}\right)du = -\frac{1}{2}\log_e|u|+C$

$= -\frac{1}{2}\log_e|\cos(2x)|+C = \boxed{\frac{1}{2}\log_e|\sec(2x)|+C}$

- e.g. 3. $\int\left(\frac{1}{1-\cos(2x)}\right)dx = \int\left(\frac{1}{1-[\cos^2(x)-\sin^2(x)]}\right)dx = \int\left(\frac{1}{2\sin^2(x)}\right)dx$

$= \frac{1}{2}\int(\operatorname{cosec}^2(x))dx = \boxed{-\frac{1}{2}\cot^2(x)+C}$

- e.g. 4. $\int\left(\frac{1}{\sqrt{17+12x-2x^2}}\right)dx = \int\frac{1}{\sqrt{17-2(x^2-6x)}}dx = \int\frac{1}{\sqrt{17+18-2(x^2-6x+9)}}dx = \int\frac{1}{\sqrt{35-2(x-3)^2}}dx$

Let $u = \sqrt{2}(x-3) \rightarrow \frac{du}{dx} = \sqrt{2} \rightarrow du = \sqrt{2}\,dx$

$\rightarrow \int\frac{1}{\sqrt{35-2(x-3)^2}}dx = \frac{1}{\sqrt{2}}\int\frac{1}{\sqrt{35-u^2}}du = \frac{\sqrt{2}}{2}\arcsin\left(\frac{u}{\sqrt{35}}\right)+C$

$= \frac{\sqrt{2}}{2}\arcsin\left(\sqrt{2}(x-3)\times\frac{\sqrt{35}}{35}\right)+C = \boxed{\frac{\sqrt{2}}{2}\arcsin\left(\frac{\sqrt{70}(x-3)}{35}\right)+C}$

(SPLIT technique)

- e.g. 5. Find an antiderivative of $\frac{4+x}{\sqrt{1-3x^2}}$

Let $I_1 = \int\frac{4}{\sqrt{1-3x^2}}dx$ and $I_2 = \int\frac{x}{\sqrt{1-3x^2}}dx$, where $I = I_1 + I_2$

→ Let $u = \sqrt{3}x \rightarrow \frac{du}{dx} = \sqrt{3} \rightarrow du = (\sqrt{3})\,dx$

$I_1 = \frac{4}{\sqrt{3}}\int\frac{1}{\sqrt{1-u^2}}du = \frac{4\sqrt{3}}{3}\arcsin(u)+C = \frac{4\sqrt{3}}{3}\arcsin(\sqrt{3}x)$

→ Let $w = 1-3x^2 \rightarrow \frac{dw}{dx} = -6x \rightarrow dw = (-6x)\,dx$

$I_2 = -\frac{1}{6}\int\frac{1}{\sqrt{w}}dw = -\frac{1}{3}\sqrt{w}+d = -\frac{1}{3}\sqrt{1-3x^2}+d$

$\therefore I = -\frac{1}{3}\sqrt{1-3x^2}+\frac{4\sqrt{3}}{3}\arcsin(\sqrt{3}x)+C_1$

(where $C = c+d$)

Let $C_1 = 0 \rightarrow I = \boxed{-\frac{1}{3}\sqrt{1-3x^2}+\frac{4\sqrt{3}}{3}\arcsin(\sqrt{3}x)}$

Exercise 7D Definite integrals by substitution

- e.g. 1 Evaluate $\int_{\pi/2}^{3\pi/4} (\cos^2(2x)\sin(2x))\,dx$. (VCAA 2010 Exam 1 Q6)

Let $u = \cos(2x) \rightarrow \frac{du}{dx} = -2\sin(2x) \rightarrow du = -2\sin(2x)\,(dx)$

When $x = \frac{\pi}{2}$, $u = -1$, When $x = \frac{3\pi}{4}$, $u = 0$ (N.B. Don't mix variables!)

$\rightarrow \int_{\pi/2}^{3\pi/4} \left[\left(-\frac{1}{2}\cos^2(2x)\right)\left(-2\sin(2x)\right)\right] dx = -\frac{1}{2}\int_{-1}^{0} u^2\,du$ (NOT $-\frac{1}{2}\int_{\pi/2}^{3\pi/4} u^2\,du$)

$= -\frac{1}{2}\left[\frac{1}{3}u^3\right]_{-1}^{0} = -\frac{1}{2}\left(0 + \frac{1}{3}\right) = \boxed{-\frac{1}{6}}$

- e.g. 2 Evaluate $\int_0^1 e^{2x}\cos(e^{2x})\,dx$.

Let $u = e^{2x} \rightarrow \frac{du}{dx} = 2e^{2x} \rightarrow du = (2e^{2x})\,dx$

When $x = 0$, $u = 1$. When $x = 1$, $u = e^2$

$\rightarrow \int_1^{e^2} \left(\frac{1}{2}\cos(u)\right) du = \frac{1}{2}\left[\sin(u)\right]_1^{e^2}$

$\therefore = \boxed{\frac{1}{2}\left[\sin(e^2) - \sin(1)\right]}$

- e.g. 3. Evaluate $\int_0^1 \left((x+3)\sqrt{3x+1}\right) dx$.

Let $u = 3x+1 \rightarrow \frac{du}{dx} = 3 \rightarrow du = 3\,(dx)$

$\rightarrow 3x = u+1 \rightarrow x = \frac{u+1}{3} \rightarrow x+3 = \frac{u+8}{3}$

When $x = 0$ $u = 1$, When $x = 1$, $u = 4$.

$\rightarrow \int_1^4 \left(\frac{1}{3}\left(\frac{u+8}{3}\right)\sqrt{u}\right) du = \frac{1}{9}\int_1^4 \left(u^{3/2} + 8u^{1/2}\right) du$

$= \frac{1}{9}\left[\frac{2}{5}u^{5/2} + \frac{16}{3}u^{3/2}\right]_1^4$

$= \frac{1}{9}\left[\frac{2}{5}(32) + \frac{16}{3}(8) - \frac{2}{5} - \frac{16}{3}\right]$

$= \frac{1}{9}\left(\frac{192}{15} + \frac{640}{15} - \frac{6}{15} - \frac{80}{15}\right)$

$= \frac{1}{9}\left(\frac{746}{15}\right) = \boxed{\frac{746}{135}}$

Normal integration answer:
$\frac{2}{45}(3x+1)^{5/2} + \frac{16}{27}(3x+1)^{3/2} + c$

- e.g. 4. Evaluate $\int_1^2 x^2(13 - 5x^3)^{1/3}\,dx$.

Let $u = 13 - 5x^3 \rightarrow \frac{du}{dx} = -15x^2 \rightarrow du = (-15x^2)\,dx$

When $x = 1$, $u = 8$. When $x = 2$, $u = -27$.

$\rightarrow \int_1^2 (13 - 5x^3)^{1/3}(x^2)\,dx = -\frac{1}{15}\int_8^{-27} u^{1/3}\,du = \frac{1}{15}\int_{-27}^{8} u^{1/3}\,du$

$= \frac{1}{15}\left[\frac{3}{4}u^{4/3}\right]_{-27}^{8} = \frac{1}{20}\left[(8)^{4/3} - (-27)^{4/3}\right]$

$= \frac{1}{20}(16 - 81) = -\frac{65}{20} = \boxed{-\frac{13}{4}}$

Exercise 7E Integration using trig identities

→ Antidifferentiation of form $\sin^m(hx)\cos^n(hx)$

① One of m or n is equal to 1 (Reverse Chain Rule)

$\int \sin(hx)\cos^n(hx)\,dx = -\frac{1}{h(n+1)}\cos^{n+1}(hx)+c$

$\int \sin^m(hx)\cos(hx)\,dx = \frac{1}{h(m+1)}\sin^{m+1}(hx)+c$

② m (odd and positive) → power of sine odd ($m=2h+1;\ h\in\mathbb{Z}$)

$\sin^{2h+1}(x) = (\sin^2(x))^h \sin(x) = (1-\cos^2(x))^h \sin(x)$ → Sub $u=\cos(x)$

③ n (odd and positive) → power of cosine odd ($n=2h+1,\ h\in\mathbb{Z}$)

$\cos^{2h+1}(x) = (\cos^2(x))^h \cos(x) = (1-\sin^2(x))^h \cos(x)$ → Sub $u=\sin(x)$

④ m+n (odd and positive) - choice between case 2 + 3

⑤ m+n (even and positive) (Double angle formulae - see Ex. 3B)

$\sin^2(ax) = \frac{1}{2}(1-\cos(2ax))$

$\cos^2(ax) = \frac{1}{2}(1+\cos(2ax))$

$\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$

- $\sin^2(ax)+\cos^2(ax)=1$
- $1+\tan^2(ax)=\sec^2(ax)$
- $1+\cot^2(ax)=\operatorname{cosec}^2(ax)$

→ Antidifferentiation trig examples

- e.g 1. Find an antiderivative of $\sec^3(x)\tan(x)$

→ $\int\left(\frac{1}{\cos^3(x)}\times\frac{\sin(x)}{\cos(x)}\right)dx = \int\frac{\sin(x)}{\cos^4(x)}\,dx$

Let $u=\cos(x)$ → $\frac{du}{dx}=-\sin(x)$ → $du=(-\sin(x))\,dx$

→ $\int\frac{\sin(x)}{\cos^4(x)}\,dx = -\int\frac{1}{u^4}\,du = \frac{1}{3u^3}+c$

$\therefore = \boxed{\frac{1}{3}\sec^3(x)+c} = \boxed{\frac{1}{3}\sec^3(x)}$ (Let $c=0$)

- e.g. 2. Evaluate $\int 3\sin^2\left(\frac{3x}{2}\right)\cos^4\left(\frac{3x}{2}\right)dx$

→ $3\int\left(\left(\sin^2\left(\frac{3x}{2}\right)\cos^2\left(\frac{3x}{2}\right)\right)\cos^2\left(\frac{3x}{2}\right)\right)dx = 3\int\left(\left(\frac{1}{2}\sin(3x)\right)^2\left(\cos^2\left(\frac{3x}{2}\right)\right)\right)dx$

$= \frac{3}{4}\int\sin^2(3x)\left[\frac{1}{2}(1+\cos(3x))\right]dx$

$= \frac{3}{8}\int\sin^2(3x)\,dx + \frac{3}{8}\int\sin^2(3x)\cos(3x)\,dx$

Let $u=\sin(3x)$ → $\frac{du}{dx}=3\cos(3x)$ → $du=3\cos(3x)\,(dx)$

→ $\frac{3}{8}\int\left[\frac{1}{2}(1-\cos(6x))\right]dx + \frac{3}{8}\int\frac{1}{3}u^2\,du$

$= \frac{3}{16}\int(1-\cos(6x))\,dx + \frac{1}{8}\int u^2\,du$

$= \frac{3}{16}x - \frac{1}{32}\sin(6x) + \frac{1}{8}\left(\frac{1}{3}u^3\right)+c$

$= \boxed{\frac{3}{16}x - \frac{1}{32}\sin(6x) + \frac{1}{24}\sin^3(3x)+c}$

- e.g. 3. Find the antiderivative of $\cos^4(2x)\sin^3(2x)$. Hence, evaluate $\int_0^{\frac{\pi}{2}}\cos^4(2x)\sin^3(2x)\,dx$.

→ $\int\cos^4(2x)\sin^2(2x)\sin(2x)\,dx$

Let $u=\cos(2x)$ → $\frac{du}{dx}=-2\sin(2x)$ → $du=-2\sin(2x)\,dx$

When $x=0$, $u=1$. When $x=\frac{\pi}{2}$, $u=-1$

$\rightarrow -\frac{1}{2}\int \left[\cos^4(2x)\cdot\left(1-\cos^2(2x)\right)\cdot(-2\sin(2x))\right]dx$

$= -\frac{1}{2}\int u^4(1-u^2)\,du = -\frac{1}{2}\int (u^4-u^6)\,du$

$= -\frac{1}{2}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right] + C = -\frac{1}{10}u^5 + \frac{1}{14}u^7 + C$

$= \boxed{-\frac{1}{10}\cos^5(2x) + \frac{1}{14}\cos^7(2x) + C}$

$\rightarrow \int_0^{\frac{\pi}{2}} \cos^4(2x)\sin^3(2x)\,dx = -\frac{1}{2}\int_1^{-1}(u^4-u^6)\,du = \frac{1}{2}\int_{-1}^{1}(u^4-u^6)\,du$

$= \frac{1}{2}\left[\frac{1}{5}u^5 - \frac{1}{7}u^7\right]_{-1}^{1} = \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7} - \frac{1}{5}(-1) + \frac{1}{7}(-1)\right)$

$= \frac{1}{2}\left(\frac{7}{35} - \frac{5}{35} + \frac{7}{35} - \frac{5}{35}\right) = \boxed{\frac{2}{35}}$

- e.g. 4. Evaluate $\int_0^{\pi/4} \cos^4(x)\,dx$.

$\rightarrow \int_0^{\pi/4}\left[\frac{1}{2}(1+\cos(2x))\right]^2 dx = \frac{1}{4}\int_0^{\pi/4}\left(1+2\cos(2x)+\cos^2(2x)\right)dx$

$= \frac{1}{4}\int_0^{\pi/4}\left[1+2\cos(2x)+\frac{1}{2}(1+\cos(4x))\right]dx$

$= \frac{1}{4}\left[\frac{3}{2}x + \sin(2x) + \frac{1}{8}\sin(4x)\right]_0^{\pi/4}$

$= \frac{1}{4}\left[\frac{3}{2}\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right) + \frac{1}{8}\sin(\pi)\right] = \boxed{\frac{3\pi}{32} + \frac{1}{4}}$

- e.g. 5. $\int_0^{\pi/8} \sec^6(2x)\,dx = \int_0^{\pi/8}(\sec^2(2x))^2\sec^2(2x)\,dx$

$= \int_0^{\pi/8}(1+\tan^2(2x))^2\sec^2(2x)\,dx$

Let $u = \tan(2x) \rightarrow \frac{du}{dx} = 2\sec^2(2x) \rightarrow du = 2\sec^2(2x)\,dx$

When $x = 0$, $u = 0$. When $x = \frac{\pi}{8}$, $u = 1$

$\rightarrow \frac{1}{2}\int_0^1 (1+u^2)^2\,du = \frac{1}{2}\int_0^1 (1+2u^2+u^4)\,du$

$= \frac{1}{2}\left[u + \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \frac{1}{2}\left(1 + \frac{10}{15} + \frac{3}{15}\right) = \frac{1}{2}\left(\frac{28}{15}\right) = \boxed{\frac{14}{15}}$

Exercise 7F Further substitution

- e.g. 1. Find $\int \frac{1}{x(1+\sqrt[3]{x})}\,dx$ by substituting $x = u^3$, where $u > 0$.

Let $x = u^3 \rightarrow \frac{dx}{du} = 3u^2 \rightarrow dx = (3u^2)\,du$ $\quad (u = \sqrt[3]{x})$

$\rightarrow \int\left(\frac{1}{u^3(1+u)} \times 3u^2\right)du = 3\int\frac{1}{u(1+u)}\,du = 3\int\left(\frac{1}{u} - \frac{1}{u+1}\right)du$

$= 3\log_e|u| - 3\log_e|u+1| + C = 3\log_e\left|\frac{u}{u+1}\right| + C$

$= 3\log_e\left|\frac{\sqrt[3]{x}}{\sqrt[3]{x}+1}\right| + C = \boxed{\log_e\left|\frac{x}{(\sqrt[3]{x}+1)^3}\right| + C}$

- e.g. 2. Find $\int \frac{5^x}{5^{x-1}+10}\,dx$

Let $u = 5^{x-1} + 10 = e^{\log_e(5^{x-1})} + 10 = e^{(x-1)\ln(5)} + 10$

$\frac{du}{dx} = \ln(5)\,e^{(x-1)\ln(5)} = \ln(5)\left(5^{x-1}\right) \rightarrow du = \frac{\ln(5)\cdot 5^x}{5}\,dx$

$\rightarrow \frac{5}{\ln(5)}\int\left(\frac{\ln(5)\cdot 5^x}{5(5^{x-1}+10)}\right)dx = \frac{5}{\ln(5)}\int\frac{1}{u}\,du$

$\therefore = \boxed{\frac{5}{\ln(5)}\log_e|5^{x-1}+10| + C}$

Exercise 7G Anti-differentiation with partial fractions

→ Steps: ① Check if degree of numerator is equal to or greater than degree of denominator
If necessary, perform polynomial long division.
② Factorise the denominator
③ Express rational function as a sum of partial fractions
(if degree of numerator is one degree less than denominator)
④ Anti-differentiate each partial fraction term.

→ Partial fractions (from step 3)

Factor in denominator	Term(s) that factor contributes to partial fraction form
Non-repeated linear factor $(x-a)$	$\frac{A}{(x-a)}$
n-fold repeated linear factor $(x-a)^n$	$\frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$
non-repeated irreducible quadratic (ax^2+bx+c)	$\frac{Bx+C}{(ax^2+bx+c)}$

→ Summary of resolving proper fraction into partial fractions
- Distinct linear factors. $\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{cx+d} + \frac{B}{ex+f}$
- Repeated linear factor. $\frac{ax+b}{(cx+d)(ex+f)^2} = \frac{A}{cx+d} + \frac{B}{ex+f} + \frac{C}{(ex+f)^2}$
- Irreducible quadratic factor. $\frac{ax+b}{(cx+d)(ex^2+f)} = \frac{A}{cx+d} + \frac{Bx+C}{ex^2+f}$

→ Partial fractions examples

e.g. 1. Find $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)}\,dx$, expressing your answer in form $\log_e\left(\sqrt{\frac{a}{b}}\right)$, where a and b are positive integers. (VCAA 2017 Exam 1 Q2)

→ $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$ → $1 = A(x^2+1) + x(Bx+C)$

$1 = (A+B)x^2 + Cx + A$ → $A = 1,\ B = -1,\ C = 0$

→ $\int_1^{\sqrt{3}} \frac{1}{x(1+x^2)}\,dx = \int_1^{\sqrt{3}} \left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx$

$= \left[\log_e|x| - \frac{1}{2}\log_e(x^2+1)\right]_1^{\sqrt{3}}$

$= \log_e(\sqrt{3}) - \frac{1}{2}\log_e(4) - \log_e(1) + \frac{1}{2}\log_e(2) = \log_e\left(\sqrt{3} \times \frac{\sqrt{2}}{2}\right) = \log_e\left(\sqrt{3} \times \frac{1}{\sqrt{2}}\right)$

$\therefore = \boxed{\log_e\left(\sqrt{\frac{3}{2}}\right)}$

e.g. 2. Find an antiderivative of $\frac{1+x}{9-x^2}$, $x \in R \setminus \{-3, 3\}$. (VCAA 2011 Exam 1 Q11)

→ $\frac{1+x}{9-x^2} = \frac{A}{3+x} + \frac{B}{3-x}$ → $1+x = A(3-x) + B(3+x)$

Sub $x = -3$, $-2 = 6A$ → $A = -\frac{1}{3}$

Sub $x = 3$, $4 = 6B$ → $B = \frac{2}{3}$

$\rightarrow \int \frac{1+x}{9-x^2}\,dx = \int\left(\frac{-1}{3(3+x)} + \frac{2}{3(3-x)}\right)dx$

$= -\frac{1}{3}\log_e|3+x| - \frac{2}{3}\log_e|3-x| + c$ ← (Acceptable solution also!) (Let $c=0$)

$= -\frac{1}{3}\left(2\log_e|3-x| + \log_e|3+x|\right) + c$

$\therefore = \boxed{-\frac{1}{3}\log_e\left((3-x)^2|3+x|\right)+c} = \boxed{-\frac{1}{3}\log_e\left(|3-x|^2|3+x|\right)}$ (Let $c=0$)

· e.g. 3. Find $\int \frac{x^5+2}{x^2-1}\,dx$.

① :

$$\begin{array}{r|l} & x^3+x \\ \hline x^2-1 & x^5+2 \\ & x^5-x^3 \\ \hline & x^3+2 \\ & x^3-x \\ \hline & x+2 \end{array}$$

② : $\frac{x^5+2}{x^2-1} = x^3+x+\frac{x+2}{x^2-1}$

Now $\frac{x+2}{x^2-1} = \frac{A}{x+1} + \frac{B}{x-1}$

$x+2 = A(x-1) + B(x+1)$

Sub $x=-1$, $1 = -2A \rightarrow A = -\frac{1}{2}$

Sub $x=1$, $3 = 2B \rightarrow B = \frac{3}{2}$

$\rightarrow \frac{x+2}{x^2-1} = \frac{-1}{2(x+1)} + \frac{3}{2(x-1)}$

③ : $\int\left(x^3+x+\frac{1}{2(x+1)}+\frac{3}{2(x-1)}\right)dx = \frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{1}{2}\log_e|x+1| + \frac{3}{2}\log_e|x-1| + c$

$\therefore = \boxed{\frac{1}{4}x^4 + \frac{1}{2}x^2 + \log_e\left|\frac{(x-1)^{3/2}}{(x+1)^{1/2}}\right| + c}$

· e.g. 4. Find $\int_0^1 \left(\frac{4x+1}{x^2+x+1}\right)dx$

$\rightarrow I = \int_0^1 \left(\frac{4x+2}{x^2+x+1}\right)dx - \int_0^1 \left(\frac{1}{x^2+x+1}\right)dx = I_1 - I_2$

$I_1 = 2\int_0^1 \left(\frac{2x+1}{x^2+x+1}\right)dx = \left[2\log_e|x^2+x+1|\right]_0^1 = 2\log_e(3)$

$I_2 = \int_0^1 \left(\frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)dx = \frac{2}{\sqrt{3}}\int_0^1 \left(\frac{\sqrt{3}/2}{(x+\frac{1}{2})^2+\frac{3}{4}}\right)dx = \frac{2\sqrt{3}}{3}\left[\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]_0^1$

$= \frac{2\sqrt{3}}{3}\left(\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(\frac{\sqrt{3}}{3}\right)\right) = \frac{2\sqrt{3}}{3}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{9}\pi$

$\therefore \int_0^1 \left(\frac{4x+1}{x^2+x+1}\right)dx = \boxed{2\log_e(3) - \frac{\sqrt{3}}{9}\pi}$

· e.g. 5. Find exact values of h and p such that $\int_h^4 \frac{1}{x^3-2x^2+x}\,dx = \frac{8}{3} - \log_e(p)$.

$\rightarrow \frac{1}{x^3-2x^2+x} = \frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$1 = A(x-1)^2 + Bx(x-1) + Cx$

$= (A+B)x^2 + (-2A-B+C)x + A \rightarrow A=1,\ B=-1,\ C=1$

$\rightarrow \int_h^4 \left(\frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2}\right)dx = \left[\log_e\left|\frac{x}{x-1}\right| - \frac{1}{x-1}\right]_h^4$

$= \log_e\left(\frac{4}{3}\right) - \frac{1}{3} - \log_e\left|\frac{h}{h-1}\right| + \frac{1}{h-1} = \log_e\left|\frac{4(h-1)}{3h}\right| + \left(\frac{1}{h-1} - \frac{1}{3}\right)$

→ Solve $\frac{1}{h-1} - \frac{1}{3} = \frac{8}{3}$, $\frac{1}{h-1} = 3 \rightarrow h-1 = \frac{1}{3} \rightarrow \boxed{h = \frac{4}{3}}$

→ Solve $\log_e\left(\frac{1}{p}\right) = \log_e\left|\frac{4(h-1)}{3h}\right|$ when $h = \frac{4}{3}$,

$\frac{1}{p} = \frac{4/3}{4} \rightarrow \frac{4}{3}p = 4 \rightarrow \boxed{p=3}$

Manipulation → Recognition → Isolation

Exercise 7H Further techniques and miscellaneous exercises

• e.g. 1. Find the derivative of $(x^2+\frac{1}{x})^{1/2}$. Hence, evaluate $\int_{-1}^{2}(2x-\frac{1}{x^2})(x^2+\frac{1}{x})^{-1/2}\,dx$.

→ Let $y=(x^2+\frac{1}{x})^{1/2}$ $\frac{dy}{dx}=\frac{1}{2}(x^2+\frac{1}{x})^{-\frac{1}{2}}(2x-\frac{1}{x^2})$ (chain rule)

$\frac{1}{2}\int(2x-\frac{1}{x^2})(x^2+\frac{1}{x})^{-1/2}\,dx=(x^2+\frac{1}{x})^{1/2}+C$

$\int(2x-\frac{1}{x^2})(x^2+\frac{1}{x})^{-1/2}\,dx=2(x^2+\frac{1}{x})^{1/2}+C$

$\int_{-1}^{2}(2x-\frac{1}{x^2})(x^2+\frac{1}{x})^{-1/2}\,dx=\left[2(x^2+\frac{1}{x})^{1/2}\right]_{-1}^{2}$

$=2\left[(4+\frac{1}{2})^{1/2}-(1-1)^{1/2}\right]=2(\frac{9}{2})^{1/2}=2(\frac{3}{\sqrt{2}})=\boxed{3\sqrt{2}}$

• e.g. 2. Find an antiderivative of $3\sin(2x)\,e^{\sin^2(x)}$.

→ Let $u=\sin^2(x)$ → $\frac{du}{dx}=2\sin(x)\cos(x)=\sin(2x)$ → $du=\sin(2x)\,dx$

$\int[(3e^{\sin^2(x)})\times\sin(2x)]\,dx=\int 3e^u\,du$

$=3e^u+C=\boxed{3e^{\sin^2(x)}+C}=\boxed{3e^{\sin^2(x)}}$ (Let $C=0$)

• e.g. 3. Find the derivative of $\sqrt{\cos(2x)}$. Hence evaluate $\int_0^{\frac{\pi}{6}}\sqrt{\sin(2x)\tan(2x)}\,dx$.

→ Let $y=\sqrt{\cos(2x)}$. Let $u=\cos(2x)$, so that $y=u^{1/2}$

- $\frac{du}{dx}=-2\sin(2x)$, $\frac{dy}{du}=\frac{1}{2\sqrt{u}}$

→ $\frac{dy}{dx}=\frac{-2\sin(2x)}{2\sqrt{u}}=\frac{-\sin(2x)}{\sqrt{\cos(2x)}}=\frac{-\sqrt{\sin(2x)}\sqrt{\sin(2x)}}{\sqrt{\cos(2x)}}=-\sqrt{\sin(2x)\tan(2x)}$

→ $-\int\sqrt{\sin(2x)\tan(2x)}\,dx=\sqrt{\cos(2x)}+C$

$\int\sqrt{\sin(2x)\tan(2x)}\,dx=-\sqrt{\cos(2x)}+C$

$\int_0^{\pi/6}\sqrt{\sin(2x)\tan(2x)}\,dx=-\left[\sqrt{\cos(2x)}\right]_0^{\pi/6}$

$=-\left[\sqrt{\cos(\frac{\pi}{3})}-\sqrt{\cos(0)}\right]=-\left[\sqrt{\frac{1}{2}}-1\right]=\boxed{1-\frac{\sqrt{2}}{2}}$

• e.g. 4. Find another expression for $\int\sin^3(2x)\cos^4(2x)\,dx$ after a suitable substitution.

→ $I=\int\sin(2x)\sin^2(2x)\cos^4(2x)\,dx=\int\sin(2x)(1-\cos^2(2x))\cos^4(2x)\,dx$

Let $u=\cos(2x)$ → $\frac{du}{dx}=-2\sin(2x)$ → $du=-2\sin(2x)\,dx$

→ $I=-\frac{1}{2}\int(1-\cos^2(2x))(\cos^4(2x))(-2\sin(2x))\,dx$

$=-\frac{1}{2}\int(u^4(1-u^2))\,du=\boxed{\frac{1}{2}\int(u^6-u^4)\,du}$

• e.g. 5. Find $\int\frac{x}{1-x^2+\sqrt{1-x^2}}\,dx$.

→ Let $u=1-x^2$ → $\frac{du}{dx}=-2x$ → $du=-2x\,dx$

$-\frac{1}{2}\int\frac{-2x}{1-x^2+\sqrt{1-x^2}}\,dx=-\frac{1}{2}\int\frac{1}{u+\sqrt{u}}\,du$

Let $w=\sqrt{u}$ → $u=w^2$ → $\frac{du}{dw}=2w$ → $du=2w\,dw$

→ $-\frac{1}{2}\int\frac{2w}{w^2+w}\,dw=-\int\frac{1}{w+1}\,dw$

$=-\log_e|w+1|+C=-\log_e|\sqrt{u}+1|+C$

$=\boxed{-\log_e|\sqrt{1-x^2}+1|+C}$

- e.g.6. Find $\int \sqrt{a^2-x^2}\,dx$, in terms of a $(a>0)$.

→ Let $x = a\sin(u) \rightarrow u = \sin^{-1}\left(\frac{x}{a}\right)$

$\frac{dx}{du} = a\cos(u) \rightarrow dx = a\cos(u)\,du$

$\rightarrow \int\left(\sqrt{a^2 - a^2\sin^2(u)} \times a\cos(u)\right)du = \int\left(\sqrt{a^2(1-\sin^2(u))} \times a\cos(u)\right)du$

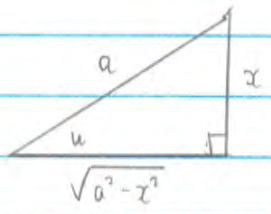
$= \int\left(a\sqrt{\cos^2(u)} \times a\cos(u)\right)du = a^2\int\cos^2(u)\,du$

$= a^2\int\left(\frac{1}{2} + \frac{1}{2}\cos(2u)\right)du = a^2\left[\frac{1}{2}u + \frac{1}{4}\sin(2u)\right] + c$

$= \frac{1}{2}a^2u + \frac{1}{4}a^2\left[2\sin(u)\cos(u)\right] + c$

$= \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^2\left(\frac{x}{a}\right)\left(\frac{\sqrt{a^2-x^2}}{a}\right) + c$

$\therefore = \boxed{\frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2} + c}$



- e.g.7. Find an antiderivative of $\frac{2x^2+1}{4x^2-4x+3}$.

→ $\frac{2x^2+1}{4x^2-4x+3} = \frac{1}{2} + \frac{2x-\frac{1}{2}}{4x^2-4x+3}$

→ $\int\frac{2x^2+1}{4x^2-4x+3}\,dx = \int\left(\frac{1}{2} + \frac{2x-\frac{1}{2}}{4x^2-4x+3}\right)dx$

$= \frac{1}{2}x + \int\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx + c_1$

Let $c_1 = 0$. → $\frac{1}{2}x + \int\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx$

$\frac{\frac{1}{2}}{4x^2-4x+3}\overline{)\,2x^2+0x+1}$

$2x^2 - 2x + \frac{3}{2}$

$2x - \frac{1}{2}$

From $\int\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx$, $\int\frac{2x-\frac{1}{2}}{4x^2-4x+3}\,dx = \int\frac{2x-1}{4x^2-4x+3}\,dx + \int\frac{\frac{1}{2}}{4x^2-4x+3}\,dx$

$= \frac{1}{4}\int\frac{8x-4}{4x^2-4x+3}\,dx + \frac{1}{2}\int\frac{1}{(2x-1)^2+2}\,dx$

Let $I_1 = \frac{1}{4}\int\frac{8x-4}{4x^2-4x+3}\,dx$ and $I_2 = \frac{1}{2}\int\frac{1}{(2x-1)^2+2}\,dx$

① From I_1, let $u = 4x^2-4x+3 \rightarrow \frac{du}{dx} = 8x-4$

$I_1 = \frac{1}{4}\int\frac{1}{u}\,du = \frac{1}{4}\log_e|u| + c_2 = \frac{1}{4}\log_e|4x^2-4x+3| + c_2$

$I_1 = \frac{1}{4}\log_e(4x^2-4x+3)$ (Let $c_2 = 0$)

② From I_2, let $p = 2x-1 \rightarrow \frac{dp}{dx} = 2$

$I_2 = \frac{1}{2\sqrt{2}}\left(\frac{1}{2}\right)\int\frac{\sqrt{2}(2)}{(2x-1)^2+2}\,dx = \frac{1}{4\sqrt{2}}\int\frac{\sqrt{2}}{p^2+2}\,dp$

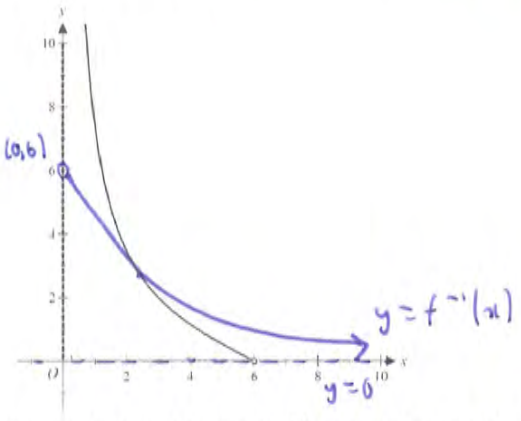
$= \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{p}{\sqrt{2}}\right) + c_3 = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{\sqrt{2}}{2}(2x-1)\right) + c_3$

→ $I_2 = \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{\sqrt{2}}{2}(2x-1)\right)$ (Let $c_3 = 0$)

→ $\int\frac{2x^2+1}{4x^2-4x+3}\,dx = \boxed{\frac{1}{2}x + \frac{1}{4}\log_e|4x^2-4x+3| + \frac{\sqrt{2}}{8}\tan^{-1}\left(\frac{\sqrt{2}}{2}(2x-1)\right)}$

Consider the function $f:\{x:0<x<6\}\to R, f(x)=2\cot\left(\frac{\pi}{12}x\right)$.

A part of the graph of f is shown below.



a. On the set of axes above, sketch the graph of f^{-1}, the inverse function of f. Label the endpoint of f^{-1} with its coordinates and any asymptote with its equation. 2 marks

b. The rule of f^{-1} can be expressed as $f^{-1}(x)=k\tan^{-1}\left(\frac{2}{x}\right)$, where k is a positive real constant.

Find the exact value of k. 1 mark

As $x\to 0$, $\tan^{-1}\left(\frac{2}{x}\right)\to\frac{\pi}{2}$

$k\tan^{-1}\left(\frac{2}{x}\right)\to 6$

$\therefore k=6\times\frac{2}{\pi}\to \boxed{k=\frac{12}{\pi}}$

c. Show, using calculus, that the derivative of $\log_e\left(x^2+4\right)+x\tan^{-1}\left(\frac{2}{x}\right)$ is $\tan^{-1}\left(\frac{2}{x}\right)$. 3 marks

$\frac{d}{dx}\left(\log_e(x^2+4)\right)=\frac{2x}{x^2+4}$

$\frac{d}{dx}\left(x\tan^{-1}\left(\frac{2}{x}\right)\right)=x\times\frac{-2}{x^2+4}+\tan^{-1}\left(\frac{2}{x}\right)$

$\therefore \frac{d}{dx}\left(\log_e(x^2+4)+x\tan^{-1}\left(\frac{2}{x}\right)\right)$

$=\frac{2x}{x^2+4}-\frac{2x}{x^2+4}+\tan^{-1}\left(\frac{2}{x}\right)$

$=\tan^{-1}\left(\frac{2}{x}\right)$ (shown)

d. Hence, show using calculus that the area of the region bounded by the graph of f^{-1} and the x-axis for $0<x\leq 2$ is $\frac{12\log_e(2)+6\pi}{\pi}$. 3 marks

$\text{Area}=\lim_{a\to 0}\frac{12}{\pi}\int_a^2\tan^{-1}\left(\frac{2}{x}\right)dx$

$=\lim_{a\to 0}\frac{12}{\pi}\left[\log_e(x^2+4)+x\tan^{-1}\left(\frac{2}{x}\right)\right]_a^2$

$=\frac{12}{\pi}\left[\left(\log_e(8)+2\tan^{-1}(1)\right)-\left(\log_e(4)+0\times\frac{\pi}{2}\right)\right]$

$=\frac{12}{\pi}\left(3\log_e(2)+2\left(\frac{\pi}{4}\right)-2\log_e(2)\right)$

$=\frac{12}{\pi}\left(\log_e(2)+\frac{\pi}{2}\right)$

$=\boxed{\frac{12\log_e(2)+6\pi}{\pi}}$ (shown)

The curve shown below can be defined parametrically by (Ex 8D)

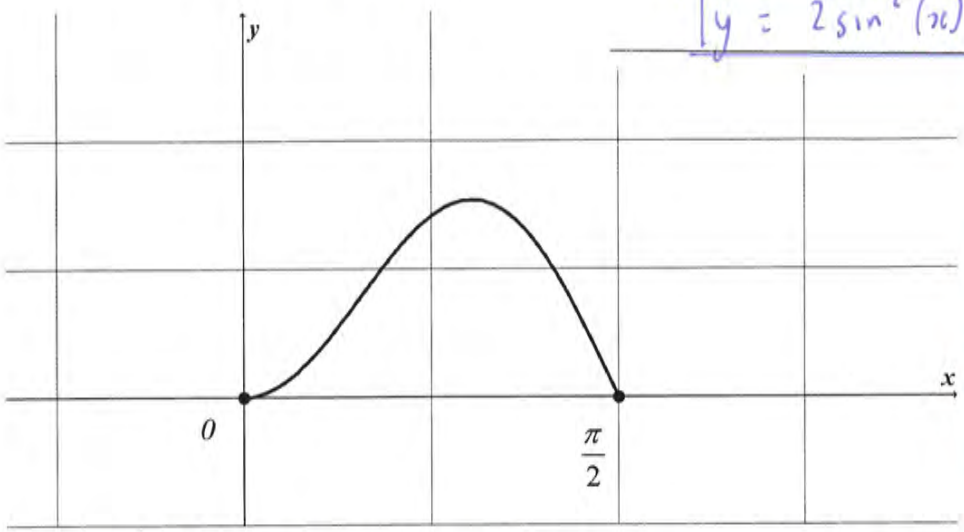
$x = \arcsin(t)$ → $t = \sin(x)$

$y = 2t^2\sqrt{1-t^2}$, where $0 \le t \le 1$.

a. Find the cartesian equation of the curve in the form $y = f(x)$. (1 mark)

$y = 2\sin^2(x)\sqrt{1-\sin^2(x)}$

$y = 2\sin^2(x)\cos(x)$



The area bounded by the curve and the x-axis is rotated about the x-axis.

b. Find the volume of the solid generated. 4 marks

$V = \pi\int_0^{\pi/2} y^2\,dx = \pi\int_0^{\pi/2} 4\sin^4(x)\cos^2(x)\,dx$

$= \pi\int_0^{\pi/2} (4\sin^2(x)\cos^2(x))\sin^2(x)\,dx$

$= \pi\int_0^{\pi/2} (2\sin(x)\cos(x))^2\sin^2(x)\,dx$

$= \pi\int_0^{\pi/2} \sin^2(2x)\left(\frac{1}{2} - \frac{1}{2}\cos(2x)\right)dx$

$= \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\,dx - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

$= \frac{\pi}{2}\int_0^{\pi/2} \frac{1}{2}(1-\cos(4x))\,dx - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

$= \frac{\pi}{4}\int_0^{\pi/2} (1-\cos(4x))\,dx - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

$= \frac{\pi}{4}\left[x - \frac{1}{4}\sin(4x)\right]_0^{\pi/2} - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

$= \frac{\pi}{4}\left(\frac{\pi}{2} - 0\right) - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

$= \frac{\pi^2}{8} - \frac{\pi}{2}\int_0^{\pi/2} \sin^2(2x)\cos(2x)\,dx$

Let $u = \sin(2x)$. → $\frac{du}{dx} = 2\cos(2x)$ → $du = 2\cos(2x)\,dx$

when $x = 0$, $u = 0$. when $x = \frac{\pi}{2}$, $u = 0$.

$\therefore V = \frac{\pi^2}{8} - \frac{\pi}{4}\int_0^0 u^2\,du$

$= \boxed{\frac{\pi^2}{8} \text{ units}^3}$