UNIT 3 & 4 MATHEMATICAL METHODS

TECHNIQUES IN DIFFERENTIATION

SHORT ANSWER QUESTIONS – TEST 1

Instructions

- Answer **all** questions in the spaces provided.
- In all questions where a numerical answer is required, an exact value must be given unless otherwise specified.
- In questions where more than one mark is available, appropriate working **must** be shown.
- Unless otherwise indicated, the diagrams in this book are **not** drawn to scale.

QUESTION 1

Find the gradient function of $f: R^+ \cup \{0\} \to R$ where $f(x) = \sqrt{x}$.

3 marks

QUESTION 2

(a) Differentiate $f(x) = \dfrac{x^2(x-2)}{3x+4}$.

(b) Find $\lim_{\theta \to 0}\left(\dfrac{8\sin^2\theta\cos^2\theta}{\theta^2}\right)$.

2 + 2 = 4 marks

QUESTION 3

(a) Find $f'(3e^{\sin x})$.

(b) Find $\lim_{x \to 1} \frac{(1+h)^3 - 1}{h}$.

3 + 1 = 4 marks

QUESTION 4

(a) Find $f'(\sin x.\log_e x)$.

(b) Find $\lim_{x \to 1}\left(\frac{x-1}{x^2+x-2}\right)$.

3 + 1 = 4 marks

QUESTION 5

Find $f'(\pi)$ given that $f(x) = \sin x . e^{\sin x}$.

3 marks

QUESTION 6

(a) Let $f(x)$ and $g(x)$ be functions such that $f(2)=5$, $f'(2)=3$, $g(2)=-1$ and $g'(2)=2$. If $h(x)=\frac{g(x)}{f(x)}$, find the value of $h'(0)$.

(b) Find the derivative given that $y=x\left(\cos x-\frac{1}{e^x}\right)$.

3 + 2 = 5 marks

QUESTION 7

Given that $f : R \setminus \{0\} \to R,\ f(x) = \frac{1}{x} + 1$

$g : R \setminus \left\{\frac{1}{2}\right\} \to R,\ g(x) = \frac{1}{2x-1}$

find, if it exists:

(a) $g'(f(x))$

(b) $f'(g(x))$

1 + 2 = 3 marks

QUESTION 8

(a) Find $f'(x)$ given that $f(x)=\begin{cases}\dfrac{1}{\sqrt{x}}, & x\ge\dfrac{1}{4}\\ 3-4x, & x<\dfrac{1}{4}\end{cases}$.

(b) Hence find

(i) $f'\left(\dfrac{1}{4}\right)$

(ii) $f'(1)$

2 + 2 = 4 marks

QUESTION 9

Find $\frac{dy}{dx}$ given that $y = 3x + 2\cos x$, and hence show that y increases as x increases.

3 marks

QUESTION 10

Given that $f(x) = ax^2 + \frac{b}{x} - 1$ and $f'(-2) = 2$ and $f'(1) = -3$, find the values of a and b and hence state the equation for $f(x)$.

4 marks

SOLUTIONS

QUESTION 1

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

Rationalising the numerator:

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h}\right) \times \left(\frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$= \lim_{h \to 0} \frac{x+h-x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \left(\frac{1}{\sqrt{x+h} + \sqrt{x}}\right)$$

$$\therefore f'(x) = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}, \quad x \neq 0$$

QUESTION 2

(a)

$$f(x) = \frac{x^2(x-2)}{3x+4} = \frac{x^3-2x^2}{3x+4}$$

$$f'(x) = \frac{(3x+4)(3x^2-4x) - (x^3-2x^2)(3)}{(3x+4)^2}$$

$$= \frac{9x^3+12x^2-12x^2-16x-3x^3+6x^2}{(3x+4)^2}$$

$$= \frac{6x^3+6x^2-16x}{(3x+4)^2}$$

$$= \frac{2x(3x^2+3x-8)}{(3x+4)^2}$$

(b)

$$\lim_{\theta \to 0}\left(\frac{8\sin^2\theta\cos^2\theta}{\theta^2}\right) = \lim_{\theta \to 0} 8\left(\frac{\sin\theta}{\theta}\right)^2\cos^2\theta$$

As $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$

$$= 8 \times 1^2 \times 1^2$$

$$= 8$$

QUESTION 3

(a)

Let $y = f\left(3e^{\sin x}\right)$

Let $u = 3e^{\sin x}$ Let $y = f(u)$

$\frac{du}{dx} = 3\cos x\, e^{\sin x}$ $\frac{dy}{du} = f'(u)$

$\therefore \frac{dy}{dx} = 3\cos x\, e^{\sin x} \cdot f'(u)$

$= 3\cos x\, e^{\sin x} f'\left(3e^{\sin x}\right)$

(b)

$\lim_{h \to 0} \frac{(1+h)^3 - 1}{h}$ $a^3 - b^3 = (a-b)(a^2+ab+b^2)$

$= \lim_{h \to 0} \frac{(1+h-1)\left((1+h)^2 + (1+h)(1) + (1)^2\right)}{h}$

$= \lim_{h \to 0} \frac{h\left(1+2h+h^2+1+h+1\right)}{h}$

$= \lim_{h \to 0} \left(h^2 + 3h + 3\right)$

$= 3$

QUESTION 4

(a)

Let $y = f(\sin x . \log_e x)$

Let $u = \sin x . \log_e x$ $\qquad y = f(u)$

$\frac{du}{dx} = \frac{\sin x}{x} + \cos x \log_e x \qquad \frac{dy}{du} = f'(u)$

$\therefore \frac{dy}{dx} = \left(\frac{\sin x}{x} + \cos x \log_e x\right) f'(u)$

$= \left(\frac{\sin x}{x} + \cos x \log_e x\right) f'(\sin x . \log_e x)$

(b)

$\lim_{x \to 1}\left(\frac{x-1}{x^2+x-2}\right) = \lim_{x \to 1}\left(\frac{x-1}{(x+2)(x-1)}\right)$

$= \lim_{x \to 1}\left(\frac{1}{x+2}\right)$

$= \frac{1}{3}$

QUESTION 5

$f(x) = \sin x \, e^{\sin x}$

$f'(x) = \sin x \frac{d}{dx}\left(e^{\sin x}\right) + e^{\sin x}\frac{d}{dx}(\sin x)$

$= \sin x \cos x \, e^{\sin x} + \cos x \, e^{\sin x}$

$= \cos x \, e^{\sin x}\left(\sin x + 1\right)$

$f'(\pi) = \cos\pi \, e^{\sin\pi}\left(\sin\pi + 1\right)$

$= (-1)\left(e^{0}\right)\left(0+1\right)$

$= -1 \times 1 \times 1$

$= -1$

QUESTION 6

(a)

$h'(x) = \dfrac{g'(x).f(x) - f'(x).g(x)}{[f(x)]^2}$

$h'(2) = \dfrac{g'(2).f(2) - f'(2).g(2)}{(f(2))^2}$

$= \dfrac{(2)(5) - (3)(-1)}{(5)^2}$

$= \dfrac{13}{25}$

(b) $y = x\left(\cos x - e^{-x}\right)$

$\frac{dy}{dx} = x \times \frac{d}{dx}\left(\cos x - e^{-x}\right) + \left(\cos x - e^{-x}\right) \times \frac{d}{dx}(x)$

$= x \times \left(-\sin x + e^{-x}\right) + \left(\cos x - e^{-x}\right)$

$= \cos x - e^{-x} - x\left(\sin x - e^{-x}\right)$

QUESTION 7

(a) $g(f(x))$ exists if $r_f \subseteq d_g$

$R \setminus \{1\} \not\subseteq R \setminus \{\frac{1}{2}\}$ $\therefore g(f(x))$ does not exist

$\therefore g'(f(x))$ does not exist

(b) $f(g(x))$ exists if $r_g \subseteq d_f$

$R \setminus \{0\} = R \setminus \{0\}$

$\therefore f(g(x))$ exists

$\therefore f(g(x))$ exists

$$f(g(x)) = \frac{1}{\left(\frac{1}{2x-1}\right)} + 1 = 2x - 1 + 1 = 2x, \quad x \neq \frac{1}{2}$$

$$\therefore f'(g(x)) = 2$$

QUESTION 8

(a) $$f'(x) = \begin{cases} -\frac{1}{2}x^{-3/2}, & x > \frac{1}{4} \\ -4, & x < \frac{1}{4} \end{cases}$$

(b) $f'\left(\frac{1}{4}\right) =$ undefined

$$f'(1) = -\frac{1}{2}(1)^{-3/2} = -\frac{1}{2}$$

QUESTION 9

$y = 3x + 2\cos x$

$\frac{dy}{dx} = 3 - 2\sin x$

If y increases as x increases, then gradient > 0

show $\frac{dy}{dx} > 0$

minimum value $= -2 + 3 = 1$

maximum value $= 2 + 3 = 5$ $\quad \therefore 1 \le \frac{dy}{dx} \le 5$

ie. gradient is always positive

$\therefore$ y increases as x increases

QUESTION 10

$f(x) = ax^2 + bx^{-1} - 1$

$f(1) = -3: \quad -3 = a + b - 1$

$\therefore \quad a + b = -2 \quad$ — ①

$f'(x) = 2ax - bx^{-2} = 2ax - \frac{b}{x^2}$

$f'(-2) = 2: \quad 2 = 2a(-2) - \frac{b}{2^2}$

$2 = -4a - \frac{b}{4} \quad$ — ②

$a + b = -2$

$4a - \frac{b}{4} = 2$

$\therefore a + b = -2$

$-16a - b = 8$ $\quad +$

$-15a = 6$

$\therefore a = -\frac{6}{15} = -\frac{2}{5}$

Substitute $a = -\frac{2}{5}$ into ①:

$-\frac{2}{5} + b = -2$

$b = -2 + \frac{2}{5} = -\frac{8}{5}$

$\therefore$ Equation is $y = -\frac{2x^2}{5} - \frac{8}{5x} - 1$