UNIT 1 & 2 MATHS METHODS

EXTENDED RESPONSE-STYLE QUESTIONS

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UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL FUNCTIONS QUESTION 1: CUBIC FUNCTION

QUESTION 1 (16 Marks)
A cubic function has the rule $f(x) = ax^3 + bx^2 + d$. The graph of the function cuts the y-axis at $y = -3$ and passes through the points $(1, 3)$ and $(7, -3)$.

a. Show that $a = -1,\ b = 7,\ d = -3$.

3 marks

b. Hence find:

(i) $f(0)$

1 mark

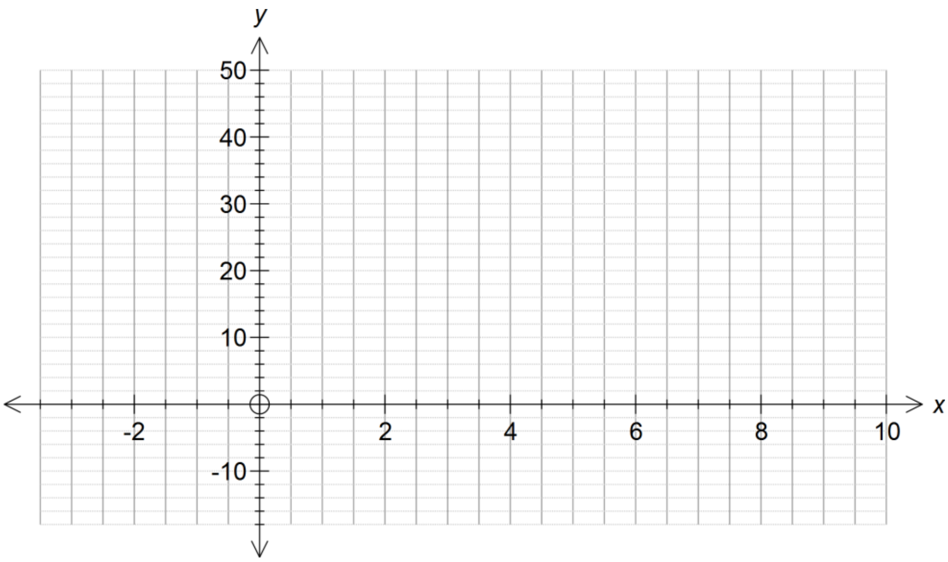
(ii) $\{x : f(x) = 3\}$

1 mark

(iii) $f\,'(1.5)$

1 mark

c. Sketch the graph of $f(x) = ax^3 + bx^2 + d$ for your values of a, b and d.



4 marks

d. Find $\{x : f(x) < 3\}$.

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1 mark

e. Is $f(x)$ a one-to-one function? Explain your answer.

---

1 mark

f. Find m such that $f_1 : [m, \infty) \to R,\ f_1(x) = ax^3 + bx^2 + d$ is a one-to-one function.

---

1 mark

g. Sketch the graph of $f_1^{-1}(x)$ for this one-to-one function on the axes in **c**.

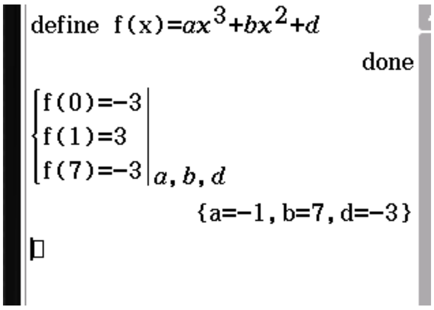
3 marks

SOLUTION

QUESTION 1

a. Set up the equations $f(0) = -3,\ f(1) = 3,\ f(7) = -3$ **2M**

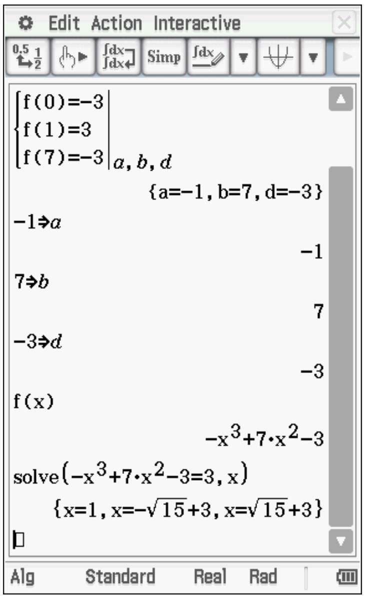
Solve simultaneously to show that $a = -1, b = 7, d = -3$ **1M**



b. Equation is $f(x) = -x^3 + 7x^2 - 3$

(i) $f(0) = -3$ **1A**

(ii) Find $\{x : f(x) = 3\}$

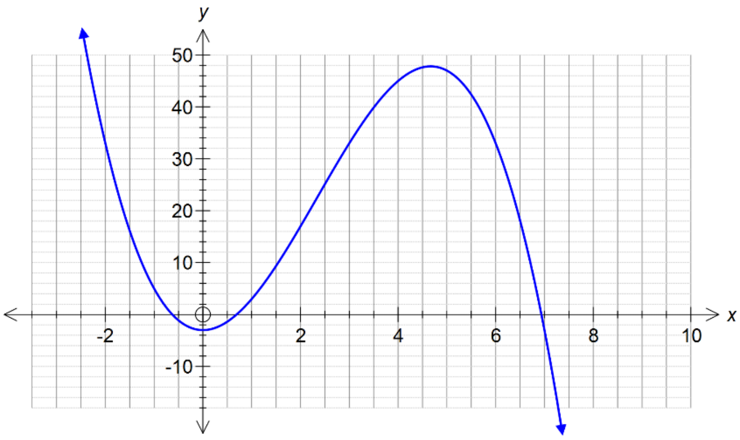


Gives $x = 1, 3 \pm \sqrt{15}$ **1A**

(iii) $f'(1.5) = \dfrac{57}{4}$ **1A**

```
define f(x)=-x^3+7x^2-3
                                  done
diff(f(x),x,1,1.5)
                                   57
                                   --
                                    4
```

c.



4A

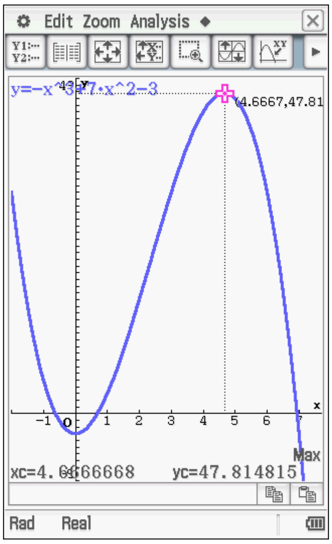
d.

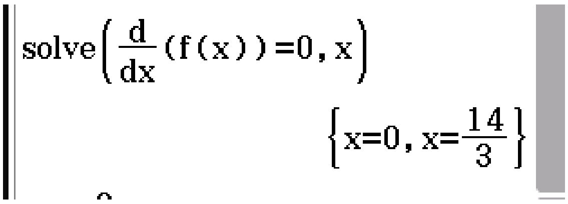
```
solve(f(x)<3,x)
        {-√15+3<x<1,√15+3<x}
```

$\{x : f(x) < 3\}$ for $\left\{-\sqrt{15}+3 < x < 1\right\} \cup \left\{x > \sqrt{15}+3\right\}$ **1A**

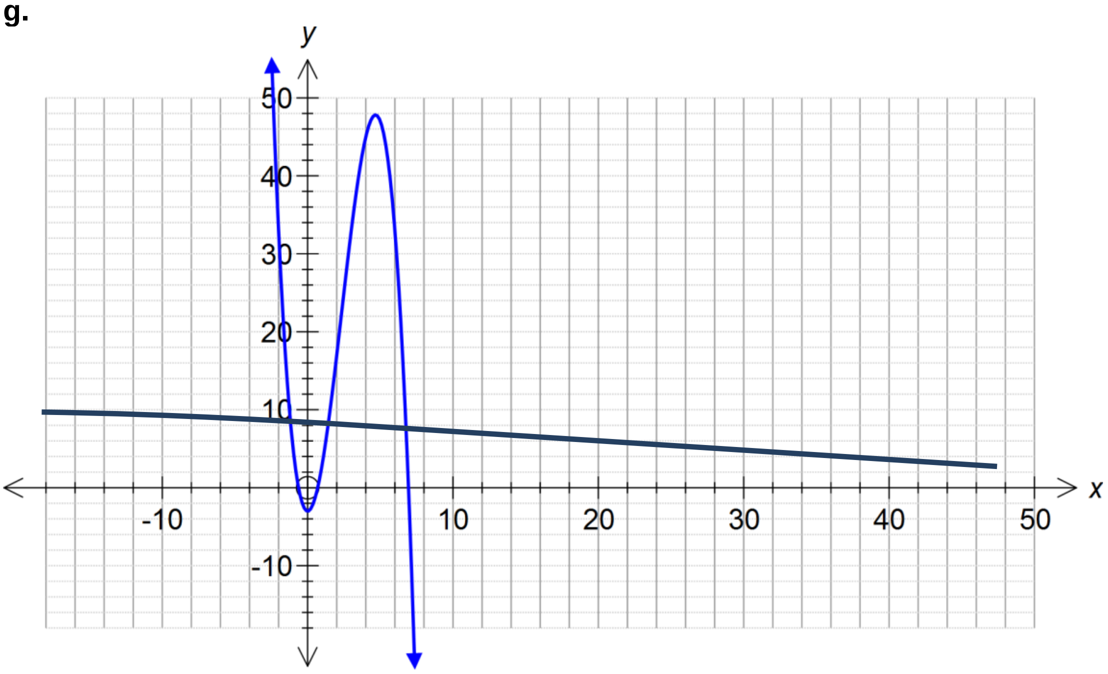
e. $f(x)$ is not a one-to-one function. It is a many-to-one function. **1A**

f. Restrict for domain $[m, \infty)$ where $m \approx 4.667$ or $m = \frac{14}{3}$. **1A**





g.

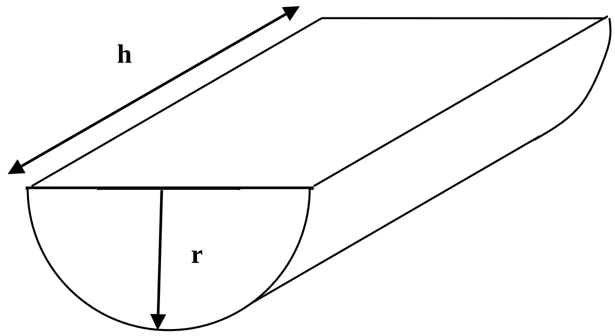


UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 2: PRECIOUS METAL

QUESTION 2 (16 Marks)



A quantity of precious metal whose volume is $500\,cm^3$ is melted and moulded into a shape of length ***h*** centimetres and uniform semi-circular cross section of radius ***r*** centimetres, as shown in the diagram.

a. Show that the total surface area, A, is $\pi r^2 + 2rh + \pi rh$.

2 marks

b. Hence show that $A = \pi r^2 + \dfrac{1000(2+\pi)}{\pi r}$.

2 marks

c. Using calculus, find the value of r, correct to 2 decimal places, for which this surface area is a minimum.

2 marks

d. Write down the minimum surface area, correct to 2 decimal places.

1 mark

To make this precious metal into a beautiful piece of jewellery, it is decided to cover the surface area with gold leaf. The flat surface area is easier to cover and costs \$***p*** per cm^2 whereas the curved surface area is more difficult to coat with gold leaf and costs \$***q*** per cm^2, where ***p*** and ***q*** are constants with ***q*** > ***p***.

e. Show that an expression for the ***total cost*** (in dollars) of covering this piece of jewellery in terms of ***r***, ***p*** and ***q*** is given by $C = \boldsymbol{p}\left(\pi r^2 + \frac{2000}{\pi r}\right) + \frac{1000}{r}\boldsymbol{q}$.

2 marks

f. Find the value of r in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ for which this cost is a minimum.

3 marks

g. Hence, or otherwise, if $\boldsymbol{p} = 15$ and $\boldsymbol{q} = 35$ write down

i. the minimum cost (correct to the nearest dollar).

1 mark

ii. the radius (in centimetres, correct to 2 decimal places).

1 mark

h. For safety reasons, the height of the jewellery piece must **not** be less than 10 centimetres in length. Find the minimum cost to cover the surface area with gold leaf, if ***p*** = 15 and ***q*** = 35. Give your answer correct to the nearest dollar.

2 marks

SOLUTION

QUESTION 2

a. Total area = Two end semi-circles + flat surface + curved surface **M1**

$$A = 2\times(\frac{1}{2}\pi\ r^2) + 2r\times h + \frac{1}{2}\times 2\pi\ r\ h$$

$$A = \pi\ r^2 + 2rh + \pi\ r\ h$$ **A1**

b. Volume = $500 = \frac{1}{2}\pi\, r^2 h$ and so $h = \frac{1000}{\pi\, r^2}$ **A1**

$$A = \pi\ r^2 + 2rh + \pi\ r\ h$$

$$= \pi\ r^2 + (2r + \pi\ r\ )h$$

$$= \pi\ r^2 + (2r + \pi\ r\ )\times\frac{1000}{\pi\ r^2}$$ **M1**

$= \pi\ r^2 + (2 + \pi\ )r\times\frac{1000}{\pi\ r^2}$ which when cancelling the r gives

$A = \pi\, r^2 + \frac{1000(2+\pi)}{\pi\, r}$, as required.

c. $A = \pi\, r^2 + \frac{1000(2+\pi)}{\pi}\times r^{-1}$

$$\frac{dA}{dr} = 2\pi\, r - \frac{1000(2+\pi)}{\pi}\times r^{-2}$$ **H1**

$= 0$ for a minimum value

Therefore $2\pi^2 r^3 = 1000(2+\pi)$ and so $r = \left(\frac{1000(2+\pi)}{2\pi^2}\right)^{\frac{1}{3}}$ = 6.39 cm **A1**

d. 384.40 cm^2 (do not accept 384.4 cm^2) **A1**

e. $C = p(\pi\, r^2 + 2rh) + q\times\pi\, r\times\frac{1000}{\pi\, r^2} = p(\pi\, r^2 + 2r\times\frac{1000}{\pi\, r^2}) + q\times\pi\, r\times\frac{1000}{\pi\, r^2}$ **M2**

(Give a method mark for each part, curved and flat)

$$\therefore C = p\left(\pi\, r^2 + \frac{2000}{\pi\, r}\right) + \frac{1000}{r}q$$

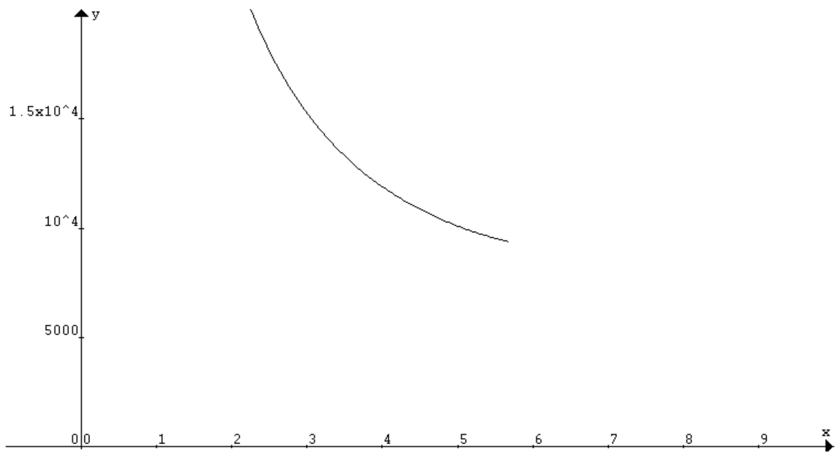
f. $\dfrac{dC}{dr} = p\left(2\pi r - \dfrac{2000}{\pi r^2}\right) - \dfrac{1000}{r^2} q$ **H1**

For minimum cost $\dfrac{dC}{dr} = 0$ and so $2\pi r\ p = \dfrac{2000p + 1000\pi q}{\pi r^2}$ **M1**

Hence $r = \left(\dfrac{1000(2p + q\pi)}{2\pi^2 p}\right)^{\frac{1}{3}}$ **A1**

g. **i.** \$8578
ii. 7.79 cm **A2**

h. Since $h = \dfrac{1000}{\pi r^2}$ then if $h = 10$, $r = \sqrt{\dfrac{100}{\pi}}$ (= 5.64 cm to 2 decimal places). **A1**



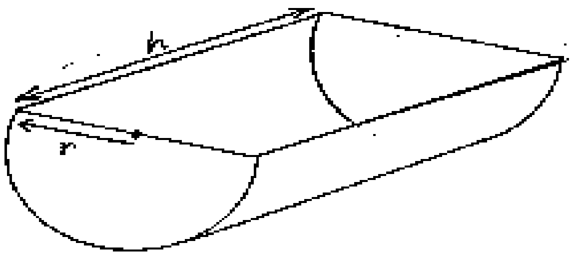
The minimum cost occurs when $r = 5.64\,cm$, and is approximately \$9396. **A1**

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 4: THIRSTY HORSE

QUESTION 4 (16 Marks)
In outback Australia in the 1850s, drums were cut in half and used for water drinking tanks for horses.



a. Show that the surface area of the drinking tank is given by $S = \pi r^2 + \pi rh$.

2 marks

b. If a fully closed drum is made of 200 m^2 sheet of metal, show that $h = \dfrac{100 - \pi r^2}{\pi r}$.

2 marks

c. Show that the volume of the drinking tank is $V = 50r - \dfrac{\pi r^3}{2}$.

2 marks

d. Using calculus, find the radius and height of the tank that gives the maximum volume. Use an appropriate technique to prove that the volume is a maximum at this radius and height.

4 marks

e. How much water can the drum hold?

1 mark

f. If the tank is being filled at 0.5 m^3 / min, how long will it take to fill the tank? Express your answer in hours and minutes.

2 marks

g. A very thirsty horse comes to drink at the tank. He drinks at an average rate of $0.8\ m^3 / \min$. State the rate at which the volume of water in the tank is changing.

1 mark

h. If the horse drinks for 20 minutes, what volume of water remains in the tank when the horse finishes drinking? State your answer correct to 2 decimal places.

2 marks

SOLUTION

QUESTION 4

a. $S.A. = \pi.r^2 + \frac{1}{2}.2\pi.r \times h = \pi.r^2 + \pi.rh$

b. $100 = \pi.r^2 + \pi.rh$

$\pi.rh = 100 - \pi.r^2$

$h = \frac{100 - \pi.r^2}{\pi.r}$

c. $V = \frac{1}{2}\pi.r^2(\frac{100 - \pi.r^2}{\pi.r})$

$V = \frac{100r}{2} - \frac{\pi.r^3}{2}$

$V = 50r - \frac{\pi.}{2}r^3$

d. $\frac{dV}{dr} = 50 - \frac{3\pi}{2}r^2 = 0$

$r^2 = \frac{100}{3\pi}$

$r = \sqrt{10.61}$ (radius must be positive in a practical situation)
$r = 3.26m$

Show maximum at r = 3.26, using a sign diagram,
When r < 3.26, then $\frac{dV}{dr} > 0$
When r > 3.26, then $\frac{dV}{dr} < 0$,
Therefore maximum.

$h = \frac{100 - \pi(3.26)^2}{\pi(3.26)}$
$h = 6.504m$

e. $V = \frac{1}{2}\pi(3.26)^2(6.504) = 108.58m^3$

f. $time = \frac{108.58}{0.5} = 217.16\min s = 3hours\ \ 37\min$

g. $-0.8m^3 / \min$

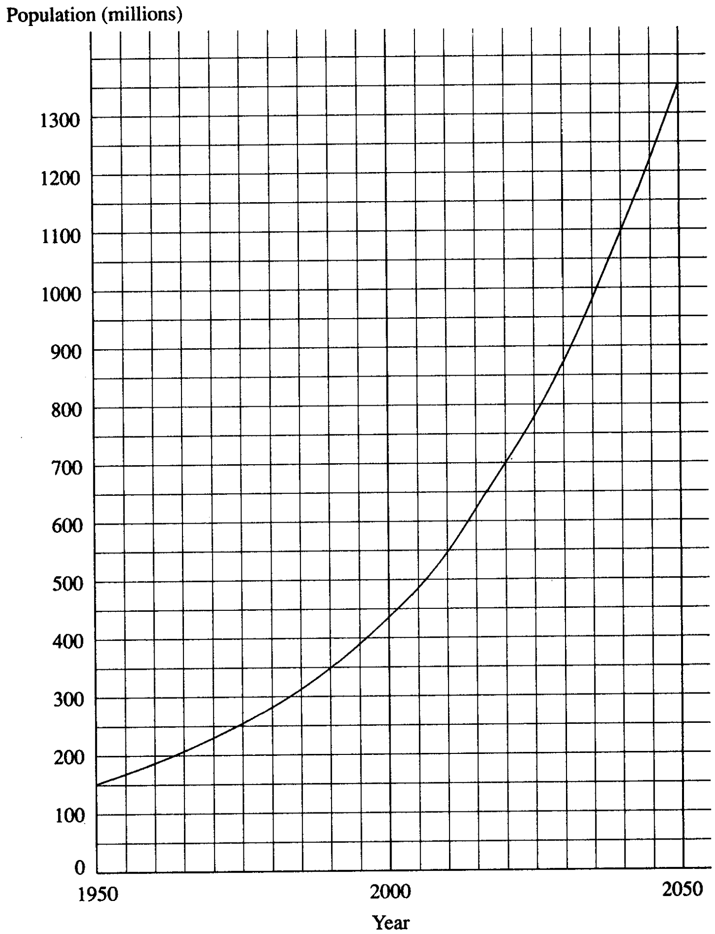
h. Volume decreases at $0.8m^3$ for 20 minutes = 0.8 x 20 = $16m^3$
Volume (after 20 mins) = 108.58 - 16 = $92.58m^3$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 5: EREHWEMOS

QUESTION 5 (17 Marks)
Erehwemos is a country whose population is growing rapidly. The below graph shows this country's population (in millions) from 1950 (when it was 150 million) to 1990 and shows what its population will be over the next 60 years if this growth continues unchecked.



a. (i) Use the graph to estimate the population of Erehwemos in 1990 and in 2020.

1 mark

(ii) Calculate the average rate of increase of population (in millions per year) from 1990 to 2020.

2 marks

b. (i) Use the graph to estimate the rate at which the population is increasing in 1990. State your answer correct to 1 decimal place.

____

____

____

2 marks

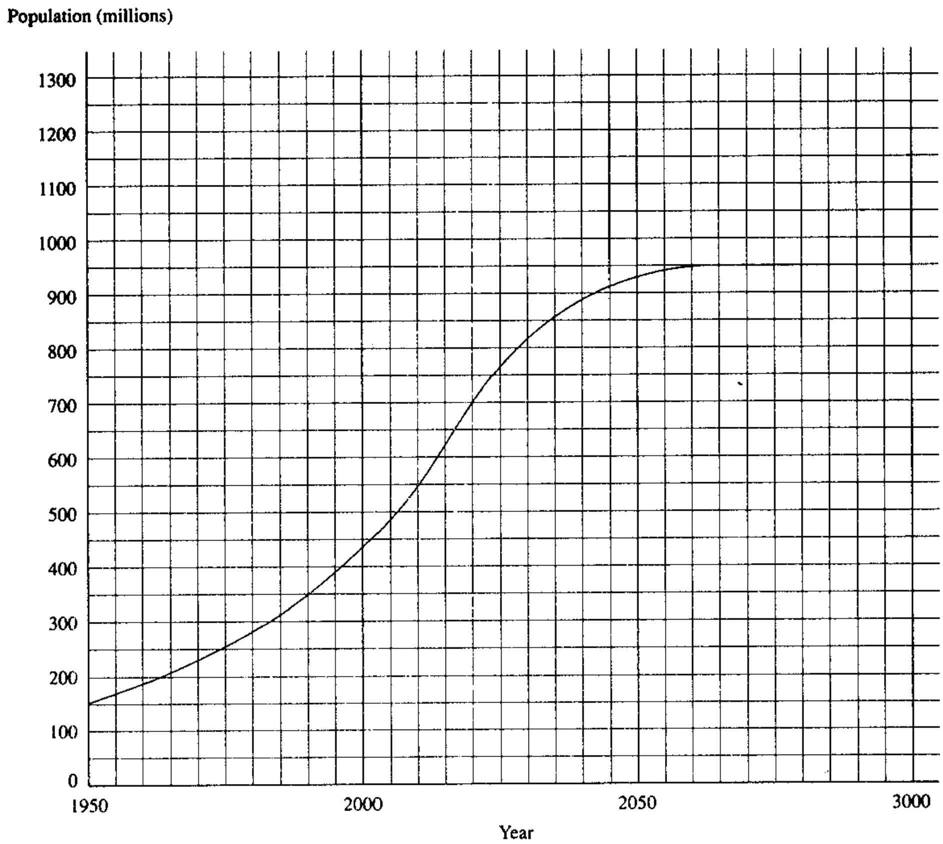
(ii) Express the rate in 1990 as a percentage of the population in 1990, to give the rate, as a percentage, at which the population is increasing in 1990. State your answer correct to 3 decimal places.

____

____

1 mark

The government of Erehwemos is developing a population control policy, which is to be introduced in 1995; one that could achieve population equilibrium (zero growth rate) some years later. The situation is shown in the graph below, which is the same as the previous graph from 1950 to 1995, but thereafter gradually changes.



c. (i) Estimate when zero growth rate will be achieved.

1 mark

(ii) Estimate when the population growth will be greatest.

1 mark

(iii) Sketch a graph showing the rate of increase of population (in millions per year) from 1950 to 2100.

3 marks

d. It has been proposed that the population control policy can be modelled by the equation $P = at^3 + bt^2 + ct + d$ where t represents the number of years from 1950 and P represents the population in millions.

(i) State the value of d.

1 mark

(ii) Write 3 equations that can be used to find the values of a, b and c.

3 marks

(iii) Hence write a matrix equation that could be used to find the values of a, b and c.

1 mark

(iv) Hence solve for the values of a, b and c. State your answers correct to 3 decimal places.

1 mark

SOLUTION

QUESTION 5

a. (i) Read values off graph

In 1990 – Pop is 350 million

In 2020 – Pop is 700 million

(ii) Av rate = m line connecting the 2 points

$(1990, 350)$ x_1 y_1

$(2020, 700)$ x_2 y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{700 - 350}{2020 - 1990}$$

$= 11\frac{2}{3}$ millions per yr

b. (i) Draw a tangent to curve at $t = 1990$ and select 2 points on the tangent and calculate gradient

$(1950, 50)$ x_1 y_1

$(2050, 800)$ x_2 y_2

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{800 - 50}{2050 - 1950} = 7.5 \text{ millions per year}$$

Answer = 7.5 millions per year.

(ii) The population in 1990 is 350 million

$\therefore$ % increase $= \frac{7.5}{350} \times 100$ = 2.143%

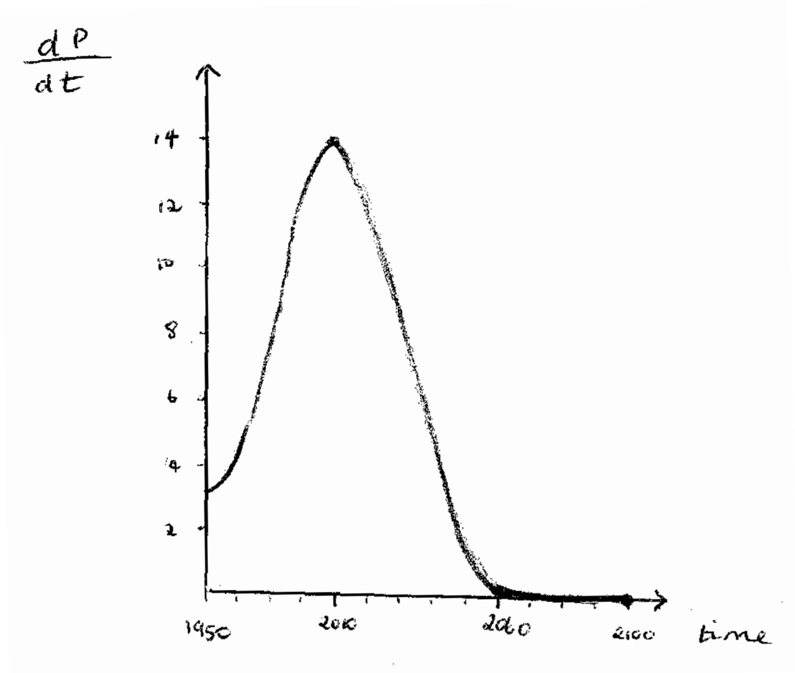
c. (i) Zero growth rate occurs when graph is flat.

i.e. In 2060.

(ii) i.e. Steepest gradient

The graph is most steep in 2010.

(iii) Draw tangents to the curve and plot.



d. (i) Substitute $(0,150)$ Answer gives $d=150$

(ii) Substitute $(60, 550)$: $216,000a+3,600b+60c+150=550$

$216,000a+3,600b+60c=400$

Substitute $(85, 850)$: $614,125a+7,225b+85c+150=850$

$614,125a+7,225b+85c=700$

Substitute $(110, 950)$: $1,331,000a+12,100b+110c+150=950$

$1,331,000a+12,100b+110c=800$

(iii) $$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 216,000 & 3,600 & 60 \\ 614,125 & 7,225 & 85 \\ 1,331,000 & 12,100 & 110 \end{bmatrix}^{-1} \begin{bmatrix} 400 \\ 700 \\ 800 \end{bmatrix}$$

Accept $$\begin{bmatrix} 216,000 & 3,600 & 60 \\ 614,125 & 7,225 & 85 \\ 1,331,000 & 12,100 & 110 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 400 \\ 700 \\ 800 \end{bmatrix}$$

(iv) $a=-0.002, b=0.356, c=-7.425$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

POLYNOMIAL & CALCULUS QUESTION 6: FUNCTIONS

QUESTION 6 (15 Marks)
Consider the graphs of the functions below:

$f(x) = x^2 + x + 2$
$g(x) = x + k$

a. (i) Without substitution, show that the solution for these equations is given by $x = \pm\sqrt{k-2}$.

2 marks

(ii) Hence find the coordinates of the point(s) of intersection of the curves when $k = 6$.

2 marks

b. (i) Using algebra, find the value(s) of k which will ensure that two points of intersection will be obtained.

2 marks

(ii) Hence find the equation of the normal at the point where curves $f(x)$ and $g(x)$ touch.

3 marks

Let A represent a point that lies on the graph of $f(x)$.

c. Use the results from **a.** and **b.** to find the equation of the tangent to the curve $f(x)$ at point A and which intersects with the line $g(x)$ at right angles.

3 marks

d. Consider the graphs of $f(x) = x^2 + x + 2$ and $h(x) = 4e^{x-1}$.

(i) Find the point(s) of intersection of the two curves.

1 mark

(ii) Although the two curves intersect, they cannot be joined smoothly. Show that a smooth join cannot be attained.

2 marks

SOLUTION

QUESTION 6

a. (i) Let $x^2+x+2 = x+k$

$$x^2+x+2-x-k=0$$
$$x^2+2-k=0$$
$$x^2=k-2$$
$$x=\pm\sqrt{k-2}$$

(ii) Sub $k=6$

$$x=\pm\sqrt{6-2} = \pm 2$$

$\therefore (-2,4)$ and $(2,8)$

b. (i) 2 solutions when $b^2-4ac>0$

$$x^2+0x+(2-k)=0$$
$$\Delta \Rightarrow \quad 0^2-4(1)(2-k) > 0$$
$$-8+4k>0$$
$$4k>8$$
$$k>2$$

(ii) Curves touch when $b^2-4ac=0$

$\therefore k=2$

$g(x) = x+k = x+2 =$ tangent

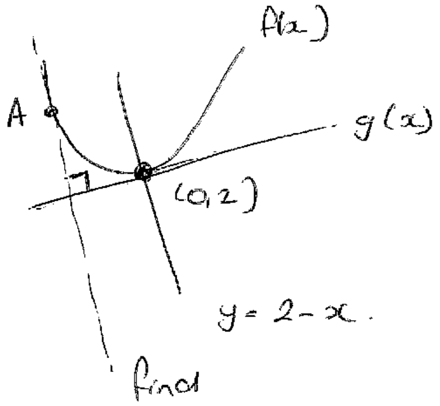
$m_{tangent} = 1 \qquad \therefore m_{normal} = -1$

when $k=2$, $\quad x=\pm\sqrt{2-2}=0$

$\therefore y=2$

$y-2=-1(x-0) \qquad \therefore y=2-x$

c.



$m_{tangent} = m_{normal}$ of $y = 2 - x$

$\therefore m = -1$

$\therefore \frac{dy}{dx}$ at $A = -1$

$y = x^2 + x + 2$

$\frac{dy}{dx} = 2x + 1 = -1$

$2x = -2$

$x = -1$

$\therefore y = (-1)^2 + (-1) + 2 = 2 \qquad (-1, 2).$

$y - 2 = -1(x + 1)$

$y = 2 - x - 1$

$y = 1 - x$

d. (i) *Let f(x) = h(x)*

$x^2 + x + 2 = 4e^{x-1}$

Intersect at $(1, 4)$

(ii) To join smoothly ⇒ gradients at point of contact must be the same.

$\therefore$ show gradients at $x = 1$ are different.

On $y = x^2 + x + 2$

$\frac{dy}{dx} = 2x + 1$

At $x = 1$, $\frac{dy}{dx} = 3$

On $y = 4e^{x-1}$

Using technology:

$\frac{dy}{dx}$ at $x = 1$ is 4

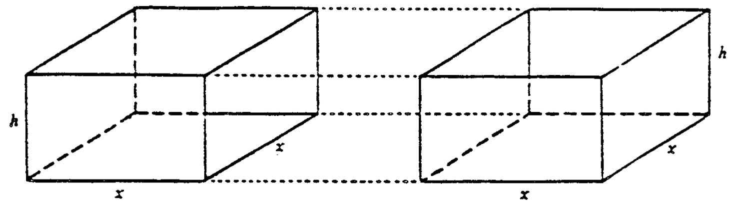
As gradients are different, join is not smooth.

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

MIXED FUNCTIONS & CALCULUS QUESTION 1: MATCHBOX

QUESTION 1 (11 Marks) **– CHALLENGING QUESTION**
A matchbox consists of a tray that is open at the top, with a square base of side $x\ cm$ and depth $h\ cm$, and a cover with open ends; into which the tray can be inserted. In order to simplify calculations, the cover may be assumed to have the same dimensions as the tray.



The cost of the material for the tray is p cents per cm^2 and the cost of the material for the cover is q cents per cm^2. If C is the cost of the material used in making a complete matchbox of volume $10\ cm^3$, then C can be written in the form $\frac{A}{x} + Bx^2$.

a. Find an expression for h in terms of x.

1 mark

b. (i) Show that the cost of the material can be written as $C = \frac{40p + 20q}{x} + x^2(p + 2q)$.

2 marks

(ii) Hence find an expression for A and B in terms p and q.

1 mark

c. Use calculus to find the value of x (in terms of p and q) for which C is a minimum.

3 marks

d. To allow the cover to slide over the tray, there must be a clearing of at least $\sqrt{3}\ \mu m$, $(\sqrt{3}\times 10^{-6} m)$ between the cover and the tray. The measure of this clearing is modelled by the equation:

$d:\{x : x \ge 2\} \to R,\ where\ d(x) = \sqrt{(x^2 - p)}$.

(i) If $d(x)$ has an inverse function $d^{-1}:\{x : x \ge 1\} \to R.$, show that $p = 3$.

2 marks

(ii) Find the equation describing $d^{-1}(x)$ and state the corresponding domain and range.

2 marks

SOLUTION

QUESTION 1

a. $V = 10 \text{ cm}^3$

$V = x^2 h$

$10 = x^2 h$

$h = \frac{10}{x^2}$ m

b. (i) cost for tray $= p$ ¢/cm²

$SA(\text{tray}) = x^2 + 4hx$

$= x^2 + 4 \times \frac{10}{x^2} \times x$

$= x^2 + \frac{40}{x}$

$\therefore \text{Cost (tray)} = p\left(x^2 + \frac{40}{x}\right)$

cost for cover $= q$ ¢/cm²

$SA(\text{cover}) = 2x^2 + 2xh$

$= 2x^2 + 2x \times \frac{10}{x^2}$

$= 2x^2 + \frac{20}{x}$

$\text{cost (cover)} = q\left(2x^2 + \frac{20}{x}\right)$

$= 2x^2 q + \frac{20q}{x}$

$\therefore \text{Total Cost} = px^2 + \frac{40p}{x} + 2x^2 q + \frac{20q}{x}$

$= \frac{40p + 20q}{x} + x^2(p + 2q)$

(ii) $C = \frac{A}{x} + Bx^2$

Equate

$A = 40p + 20q \qquad B = p + 2q$

c.

$$C = (40p + 20q)x^{-1} + x^2(p + 2q)$$

$$\frac{dC}{dx} = -\frac{(40 + 20q)}{x^2} + 2x(p + 2q)$$

$$0 = -\frac{(40p + 20q)}{x^2} + 2x(p + 2q)$$

$$0 = -40p - 20q + 2x^3(p + 2q)$$

$$2x^3 = \frac{40p + 20q}{p + 2q}$$

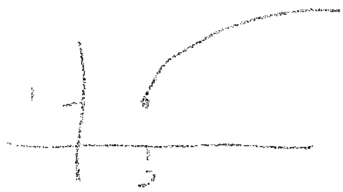
$$x^3 = \frac{40p + 20q}{2(p + 2q)}$$

$$= \frac{2(20p + 10q)}{2(p + 2q)}$$

$$x = \left(\frac{20p + 10q}{p + 2q}\right)^{1/3}$$

d. (i) d : domain $x \geq 2$

ran $x \geq 1$ (from d^{-1})



$\therefore$ When $x = 2$, $y = 1$

$1 = \sqrt{2^2 - p}$

$1 = \sqrt{4 - p}$

$1 = 4 - p$

$\therefore p = 3$

(ii) If $y = \sqrt{x^2 - 3}$

Let

$x = \sqrt{y^2 - 3}$

$x^2 = y^2 - 3$

$y^2 = x^2 + 3$

$y = \sqrt{x^2 + 3}$

$d^{-1}(x) = \sqrt{x^2 + 3}$

$x \geq 1$

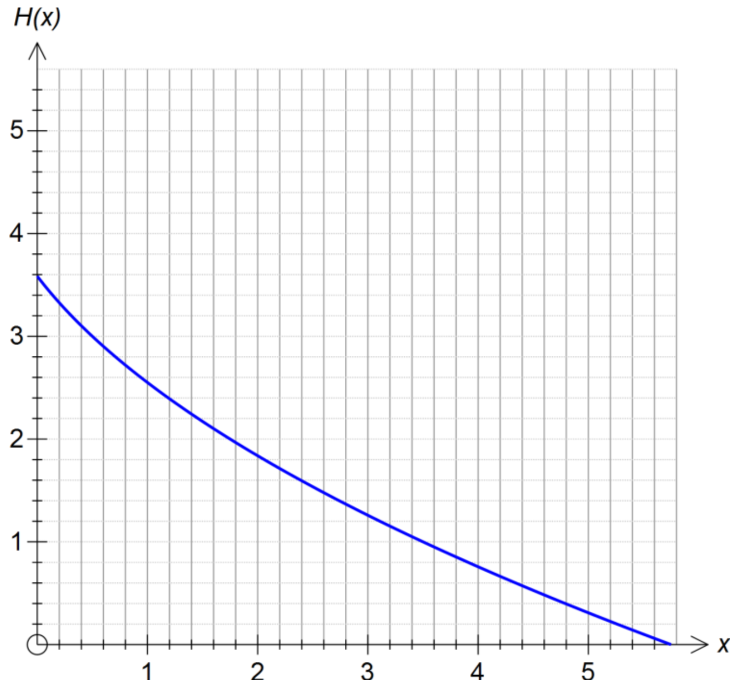
$d^{-1}(x) \geq 2$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RADICAL FUNCTIONS & CALCULUS QUESTION 2: SLIDE

QUESTION 2 (16 Marks)
The height, in metres, of a slide can be modelled by the function $H(x) = -\sqrt{4x+2}+5$ where x is the horizontal distance, in metres, from the base of the slide.



a. Find where $H(x)=0$.

____

1 mark

b. If the slide ends where it touches the ground, state an appropriate domain and range for $H(x)$.

____

____

____

2 marks

c. Find the average rate of change in the height of the slide from $x=0$ to $x=2$.

2 marks

d. State the instantaneous rate of change of the slide at $x=1$.

1 mark

e. Explain if your answer for **d**. is steeper/equal/shallower than the average rate of change in the height of the slide from $x=0$ to $x=2$.

2 marks

f. If the height of the slide has to be halved for safety concerns and then moved to the right by 5 metres to avoid some wet ground, state the new function, $H_T(x)$ in terms of x.

2 marks

g. Find the rule of inverse function H^{-1}, also stating the domain and range of H^{-1}.

4 marks

h. Sketch the inverse function H^{-1}, on the given graph.

2 marks

SOLUTION

QUESTION 2

a. Solve $H(x) = -\sqrt{4x+2} + 5 = 0$ gives $x = \dfrac{23}{4}$. **1A**

```
define f(x)=-√(4x+2)+5
                                   done
solve(f(x)=0,x)
                                {x=23/4}
```

b. Domain $\left[0, \dfrac{23}{4}\right]$, Range $\left[0, 5-\sqrt{2}\right]$ **2A**

c. Average rate of change

$$= \frac{f(2) - f(0)}{2-0}$$

$$= \frac{\sqrt{2} - \sqrt{10}}{2}$$

1M 1A

$$\frac{f(2)-f(0)}{2-0} \qquad \frac{-(\sqrt{10}-\sqrt{2})}{2}$$

d. $f'(1) = \dfrac{-\sqrt{6}}{3}$ **1A**

```
diff(f(x),x,1,1)
                                   -√6/3
```

e. Answer for **d**. is $f^{/}(1) \approx -0.816$

Answer for **c**. is ≈ -0.874 **1M**

Instantaneous rate of change at x = 1 is shallower than the average rate of change from $x = 0$ to $x = 2$.

1A

$$\frac{f(2)-f(0)}{2-0}$$

```
                 -0.8740320489
diff(f(x),x,1,1)
                 -0.8164965809
```

f. Halved $y = \frac{1}{2}\left(-\sqrt{4x+2}+5\right) = -\frac{1}{2}\sqrt{4x+2}+\frac{5}{2}$ **1A**

Then moved to the right gives $y = -\frac{1}{2}\sqrt{4(x-5)+2}+\frac{5}{2}$

Giving $H_T(x) = -\frac{1}{2}\sqrt{4x-18}+\frac{5}{2}$ **1A**

g. $H(x) = -\sqrt{4x+2}+5$

Swap:

$x = -\sqrt{4y+2}+5$ and solve

Giving:

$$5-x = \sqrt{4y+2}$$
$$\therefore (5-x)^2 = 4y+2$$
$$\therefore 4y = (5-x)^2 - 2$$
$$\therefore H^{-1}(x) = \frac{1}{4}(5-x)^2 - \frac{1}{2}$$

1M

Alternatively:

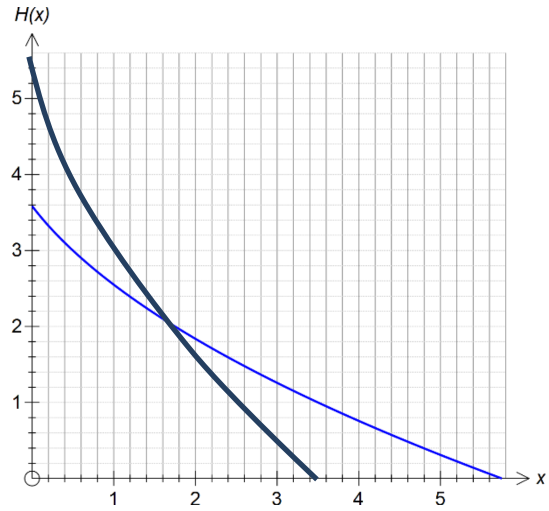
```
solve(f(y)=x,y)
```

$$\left\{y=\frac{x^2}{4}-\frac{5\cdot x}{2}+\frac{23}{4}\right\}$$

$H^{-1}(x)=\frac{x^2}{4}-\frac{5x}{2}+\frac{23}{4}$ or $H^{-1}(x)=\frac{1}{4}(5-x)^2-\frac{1}{2}$ **1A**

Domain $\left[0,5-\sqrt{2}\right]$, Range $\left[0,\frac{23}{4}\right]$ **2A**

h.



2A

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RATIONAL FUNCTIONS & CALCULUS QUESTION 1: TITANIC 2

QUESTION 1 (10 Marks)
Leonardo Di Caprio has invested in a replica of the ship Titanic, called Titanic2. Leonardo is planning a 13,000 km trip from Los Angeles to Melbourne. The cost, ***C$ per hour***, at an average speed of ***v km/hr*** is given by the equation

$$C = 0.02v^3 + \frac{240}{v}$$

a. What is the ***cost per hour*** at a speed of 15km/hr?

1 mark

b. What would be the **cost** of the 13,000km journey at 15km/hr? State your answer correct to the nearest dollar.

2 marks

c. Show that the cost of a journey of 13,000km at *v km/hr* is $260v^2 + \frac{3.12 \times 10^6}{v^2}$.

2 marks

d. (i) Leonardo wants to keep costs to a minimum. Calculate the most economical speed for the journey. State your answer correct to 2 decimal places.

2 marks

(ii) What is the minimum cost of the journey? State your answer correct to the nearest dollar.

3 marks

SOLUTION

QUESTION 1

a. $C = 0.02(15)^3 + \dfrac{240}{15} = 67.5 + 16 = \$83.50\,per.hour$ **(1 mark)**

b. $Time = \dfrac{13000}{15} = 866.67$ **(1 mark)**

$Cost = 83.50 \times 866.67 = \$72{,}367$ **(1 mark)**

c. $Time = \dfrac{13000}{v}$

$Cost = \dfrac{13000}{v} \times (0.02v^3 + \dfrac{240}{v})$ **(1 mark)**

$= \dfrac{13000}{v} \times \dfrac{2v^4 + 24000}{100v} = \dfrac{26000v^4 + 3.12 \times 10^8}{100v^2}$

$= 260v^2 + \dfrac{3.12 \times 10^6}{v^2}$ **(1 mark)**

d. (i) $\dfrac{dC}{dv} = 520v - \dfrac{6.24 \times 10^6}{v^3} = 0$ **(1 mark)**

$\dfrac{6.24 \times 10^6}{v^3} = 520v$

$6.24 \times 10^6 = 520v^4$

$v^4 = 12000$

$v = 10.47 km/hr$ **(1 mark)**

(ii) $t = \dfrac{13000}{10.4664} = 1242.076 hrs$ **(1 mark)**

$C = 0.02(10.4664)^3 + \dfrac{240}{10.4664}$

$C = 22.9309 + 22.9305 = 45.861 \$/hr$ **(1 mark)**

Total Cost = 1242.076 x 45.861 = \$56,962.85=\$56,962 (nearest dollar) **(1 mark)**

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

RATIONAL FUNCTIONS & CALCULUS QUESTION 2: HYPERBOLA

QUESTION 2 (24 Marks)

A function is defined by $f : R \setminus \{-2\} \to R$ where $f(x) = \dfrac{3x-4}{x+2}$.

a. State the domain and range of f.

2 marks

b. (i) Write $f(x)$ in the form $\dfrac{a}{x+h} + k$.

2 marks

(ii) Write f^{-1} in the form $\dfrac{a}{x+h} + k$ and state the value(s) of x for which f^{-1} is not defined.

3 marks

c. Write the equation $f(x) = x$ in the form $ax^2 + bx + c = 0$ and hence show that there are no real values of x which map themselves under the function f.

3 marks

The function g is defined by $g : R \setminus \{-2\} \to R$ where $g(x) = \dfrac{kx - 9}{x + 2}$.

d. Find the positive value of k for which there is only one value of x satisfying $g(x) = x$.

3 marks

Consider the function $f(x) = 2x^2 - 16x + 25$.

e. (i) Write $2x^2 - 16x + 25$ in the form $a[(x-b)^2 + c]$, where a, b, c are constants.

2 marks

(ii) The graph of $2x^2 - 16x + 25$ may be obtained from the graph of $y = x^2$ by means of appropriate translations and dilations. Describe suitable transformations in detail and state the order in which they would be applied.

3 marks

(iii) State the equations describing:

$y = f(x) + 2$

$y = f(x-3)$

$y = 2f(1-x)$

3 marks

f. A function f is defined for the domain $x \geq 4$ by $f(x) = 2x^2 - 16x + 25$. State the domain and range of f^{-1} and determine its equation.

3 marks

SOLUTION

QUESTION 2

a.

$$y = \frac{3x-4}{x+2},\ x \neq -2$$

dom $f = R\backslash\{-2\}$

To find the range, we solve for x:

$$(x+2)y = 3x - 4$$
$$xy + 2y = 3x - 4$$
$$2y + 4 = 3x - xy$$
$$= x(3-y)$$
$$\therefore \quad x = \frac{2y+4}{3-y},\ y \neq 3$$

$\therefore$ ran $f = R\backslash\{3\}$.

b. (i)

$$\begin{array}{r|l} & 3 \\ x+2 & 3x-4 \\ - & 3x+6 \\ \hline & -10 \end{array}$$

$$y = 3 - \frac{10}{x+2}$$

b. (ii)

$$x = \frac{-10}{y+2} + 3$$

$$x - 3 = \frac{-10}{y+2}$$

$$y + 2 = \frac{-10}{x-3}$$

$$\therefore f^{-1}(x) = \frac{-10}{x-3} - 2$$

$f^{-1}(x)$ is not defined when $x = 3$

c.

If $f(x) = x$,

then $\frac{3x-4}{x+2} = x$

$$3x - 4 = x(x+2)$$
$$3x - 4 = x^2 + 2x$$
$$-x^2 + x - 4 = 0$$
$$x^2 - x + 4 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(4)}}{2 \times 1}$$
$$= \frac{1 \pm \sqrt{1-16}}{2}$$
$$= \frac{1 \pm \sqrt{-15}}{2} \notin R$$

$\therefore$ The equation has no real roots. Hence there are no real values of x which map onto themselves under the function f.

d.

If $g(x) = x$,

then $\frac{kx-9}{x+2} = x$

$$kx - 9 = x(x+2)$$
$$= x^2 + 2x$$
$$-x^2 + kx - 2x - 9 = 0$$
$$x^2 - kx + 2x + 9 = 0$$
$$x^2 + (2-k)x + 9 = 0$$

The condition for a root is $\Delta = 0$

i.e. $b^2 - 4ac = 0$

$$(2-k)^2 - 4 \times 1 \times 9 = 0$$
$$(2-k)^2 - 36 = 0$$
$$(2-k)^2 = 36$$
$$2 - k = \pm 6$$
$$-k = \pm 6 - 2$$
$$= 4 \text{ or } -8$$
$$\therefore \quad k = -4 \text{ or } 8$$

So, the positive value of k is 8.

e. (i)

$$\begin{aligned}2x^2 - 16x + 25 &= 2(x^2 - 8x + \tfrac{25}{2})\\ &= 2(x^2 - 8x + 16 - 16 + \tfrac{25}{2})\\ &= 2[(x-4)^2 - \tfrac{32}{2} + \tfrac{25}{2}]\\ &= 2[(x-4)^2 - \tfrac{7}{2}]\end{aligned}$$

(ii) Dilation of factor 2 from the X axis.
Translation of 4 units in the positive direction along the X axis.
Translation of 7 units in the negative direction along the Y axis.

(iii)

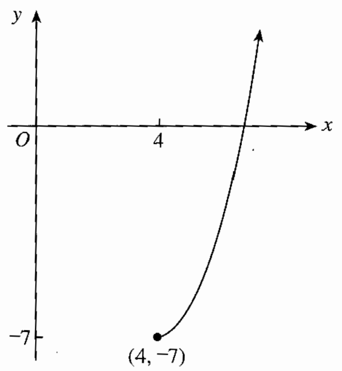
$$\begin{aligned}y = f(x)+2 &= 2x^2 - 16x + 25 + 2\\ &= 2x^2 - 16x + 27\end{aligned}$$

$$\begin{aligned}y = f(x-3) &= 2(x-3)^2 - 16(x-3) + 25\\ &= 2(x^2 - 6x + 9) - 16x + 48 + 25\\ &= 2x^2 - 12x + 18 - 16x + 48 + 25\\ &= 2x^2 - 28x + 91\end{aligned}$$

$$\begin{aligned}y = 2f(1-x) &= 2\left[2(1-x)^2 - 16(1-x) + 25\right]\\ &= 2\left[2(1 - 2x + x^2) - 16 + 16x + 25\right]\\ &= 2\left[2 - 4x + 2x^2 - 16 + 16x + 25\right]\\ &= 2\left(2x^2 + 12x + 11\right)\\ &= 4x^2 + 24x + 22\end{aligned}$$

f.

$$\begin{aligned}y &= 2x^2 - 16x + 25\\ &= 2(x-4)^2 - 7,\ x \geqslant 4,\ y \geqslant -7\end{aligned}$$



The range is $\{y : y \geqslant -7\}$.

$$\operatorname{dom} f^{-1} = \{x : x \geqslant -7\}$$
$$\operatorname{ran} f^{-1} = \{y : y \geqslant 4\}$$

The inverse of
$y = 2(x-4)^2 - 7,\ x \geqslant 4,\ y \geqslant -7$
is $x = 2(y-4)^2 - 7,\ x \geqslant -7,\ y \geqslant 4$

$$\begin{aligned}\therefore\quad x + 7 &= 2(y-4)^2\\ (y-4)^2 &= \tfrac{1}{2}(x+7)\\ y - 4 &= \pm\sqrt{\tfrac{1}{2}(x+7)}\\ y &= 4 \pm \sqrt{\tfrac{1}{2}(x+7)},\\ &= 4 + \sqrt{\tfrac{1}{2}(x+7)},\ \text{as } y \geqslant 4\end{aligned}$$

$\therefore\quad f^{-1}(x) = 4 + \sqrt{\tfrac{1}{2}(x+7)},\ x \geqslant -7.$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

TRIGONOMETRIC FUNCTIONS QUESTION 2: KING'S BEACH

QUESTION 2 (11 Marks)
The water level at King's Beach on a Monday is modelled by the equation
$h(t) = 1.5 + \sin\left(\frac{\pi t}{6}\right)$, where h is the height of the water above sea level in metres, and t represents the time in hours from 9 am in the morning.

a. Find the maximum depth of water, correct to 3 decimal places.

1 mark

b. (i) Show that the first time on Monday when the depth of the water is equal to 1.2 metres occurs at approximately $3:35\ pm$.

3 marks

(ii) If children cannot swim in the water when the depth exceeds1.2 metres, state the times when the children can safely swim on Monday. State your answer(s) correct to the nearest minute.

2 marks

c. (i) Find the average rate of change in the height of the water between 9 am and 2 pm, correct to 3 decimal places.

2 marks

(ii) Find the first 4 times when the rate of change in water level is decreasing by 0.5 metres each hour. State your answers correct to 4 decimal places.

3 marks

SOLUTION

QUESTION 2

a. Maximum Value = Amplitude + Vertical Translation

$$=1+1.5=2.500\ m$$

b. (i) $1.5+\sin\left(\dfrac{\pi t}{6}\right)=1.2$

$$\sin\left(\frac{\pi t}{6}\right)=-0.3$$

1st Quadrant Solution: $Sin^{-1}(0.3)=0.3047$

Solutions are to lie in the quadrants where sin is negative i.e. Quadrants 3 and 4.

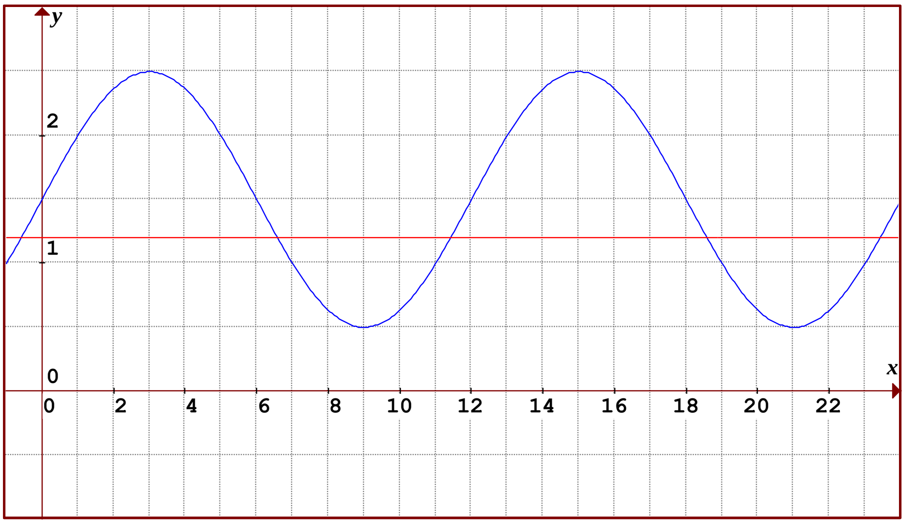
$$\frac{\pi t}{6}=\pi+0.3047,\ 2\pi-0.3047$$

$$\frac{\pi t}{6}=3.44628,\ 5.9785$$

$$\therefore t=6.582,\ 11.418\ hrs$$

First time occurs at 6.582 hours, which is at approximately 3:35am.

(ii)



Note: Monday ends at 15 hrs (midnight)

The children can safely swim on Monday between the following hours:

$\{t : 6.582 \le t \le 11.418\}$ where t represents the time in hours from 9am in the morning.

In real time: Between approximately 3:35 pm and 8:25 pm.

c. (i) The average rate of change = The gradient of the line connecting two points.

At 9am, $t = 0\ hrs$

$h(0) = 1.5 + \sin(0) = 1.5 \qquad \therefore (x_1, y_1) = (0, 1.5)$

At 2pm, $t = 5\ hrs$

$h(5) = 1.5 + \sin\left(\frac{5\pi}{6}\right) = 2 \qquad \therefore (x_2, y_2) = (5, 2)$

Average Rate of Change = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1.5}{5 - 0} = \frac{0.5}{5} = 0.100\ m/hr$

(ii) Decreasing by 0.5 metres each hour means $-0.5\ m/hr$.

Using technology:

$h'(t) = \frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right)$

Let $\frac{\pi}{6}\cos\left(\frac{\pi t}{6}\right) = -0.5$

$t = 5.4244$

$t = 6.5756$

$t = 17.4244$

$t = 18.5756$

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

EXPONENTIAL & CALCULUS QUESTION 1: XF

QUESTION 1 (15 Marks)
The X-files have discovered a new type of flesh eating micro-animal called ***XF***. The animal is captured in an air-tight container, however, if the container was to open, the number of ***XF***'s would increase exponentially.

The number of live ***XF*** after t minutes is modelled by the function $F(t) = F_o e^{kt}, t \geq 0$ where $t = 0$ is taken at the time the jar is opened,

a. There are 300 ***XF*** micro-animals in the jar before it is opened. What is the value of F_o?

1 mark

b. The container is accidentally opened, and it is found that after 10 minutes there are 700 FX micro animals. Show that $k = 0.085$.

2 marks

c. Calculate how many ***XF*** micro-animals would be alive after 1 hour.

1 mark

d. Find the rate of increase of the ***XF*** micro-animals after 1 hour.

2 marks

e. In the X-files there is information to say that one human adult can be consumed for every 1000 ***XF*** micro-animals.

i. After 1 hour, in what capacity would humans be affected?

1 mark

ii. After 2 hours, in what capacity would humans be affected?

1 mark

f. Fortunately, a laboratory has discovered a substance called **XKILL** that will kill the ***XF*** micro-animals. The formula to model this is given by $K(t) = 50e^{0.1t}$.

As soon as the **XKILL** substance is released the ***XF*** stop multiplying.

The **XKILL** substance is released 1 hour after the ***XF*** micro-animals escaped from the container.

i. How long will it take to kill half of the ***XF*** micro-animals. State your answer in hours and minutes.

2 marks

ii. How long will it take to kill ALL of the ***XF*** micro-animals? State your answer correct in hours and minutes.

1 mark

iii. How long did it take to kill the last half of the ***XF*** micro-animals?

1 mark

g. It was decided to investigate the inverse function of the ***XF*** micro-animals, to see if there would be some understanding in their structure and growth.

From the original **F(t)** function, find the inverse function and state the corresponding domain.

3 marks

SOLUTION

QUESTION 1

a. $F_o = 300$ **(1 mark)**

b. $700 = 300e^{k.10}$ **(1 mark)**

$\frac{700}{300} = e^{10k}$

$10k = \ln 2.33$

$k = 0.085$ **(1 mark)**

c. $F(t) = 300e^{0.085t}$ (1 hour = 60 mins)

$F(60) = 300e^{0.085\times 60}$

$= 49207$ **(1 mark)**

d. $F'(t) = 25.5e^{0.085t}$ **(1 mark)**

$F'(60) = 25.5e^{0.085\times 60} = 4183XF / \min$ **(1 mark)**

e. **i.** $F(60) = \frac{49207}{1000}$ from part (c)

$= 49$ adults **(1 mark)**

ii. $F(120) = 300e^{0.085(120)} = 8070956 \ (\div 1000)$

Therefore, 8071 adults **(1 mark)**

f. **i.** $F(60) = 49207$

half $24603.5 = 50e^{0.1t}$ **(1 mark)**

$492.07 = e^{0.1t}$

$0.1t = \ln 492.07$

$t = 61.99$ mins

$t = 1$ hour 2 mins **(1 mark)**

ii. $49207 = 50e^{0.1t}$

$t = 68.92$ mins

$t = 1$ hour 9 mins **(1 mark)**

iii. 7 mins **(1 mark)**

g. $F(t) = 300e^{0.085t}$

Swap F and t : $t = 300e^{0.085F}$ **(1 mark)**

$$\frac{t}{300} = e^{0.085F}$$

$$0.085F = \ln\frac{t}{300}$$

$$F = \frac{1}{0.085}\ln\frac{t}{300}$$

$F = 11.76\ln t - 67.10$, $t \geq 0$ **(2 marks)**

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

LOGARITHMIC FUNCTIONS QUESTION 1: TUNGSTEN GLOBE

QUESTION 1 (9 Marks)
The intensity (y units) of light emitted from a tungsten light globe is given by

$$y = A\log_e(t+b)$$

where t is the time in microseconds (μ sec). The initial globe intensity is 0 units and the intensity at $8\,\mu$ sec is 50 units.

a. (i) Show that $b = 1$.

1 mark

(ii) Show that $A = \dfrac{50}{\log_e 9}$.

1 mark

b. (i) Find the intensity of light at $15\,\mu$ sec.

1 mark

(ii) Find the exact time at which the intensity of the globe is $\frac{1}{\log_e 9}$ units.

2 marks

c. (i) Let $y = f(t)$. Find the equation describing the inverse function, f^{-1}.

2 marks

(ii) State the range of f^{-1}.

1 mark

d. The graph of $y = A\log_e(t+b)$ undergoes a transformation to produce the curve with equation $Y = A\log_e(t)$, where $A = \dfrac{50}{\log_e 9}$.

State the transformation required to convert $y(t)$ to $Y(t)$.

1 mark

SOLUTION

QUESTION 1

a. (i) $y = A\log_e(t+b)$

When $t=0$, $y=0$

$A\log_e(b) = 0$

$\log_e(b) = 0$
$\therefore b = 1$

(ii) $y = A\log_e(t+1)$

When $t=8$, $y=50$
$A\log_e(8+1) = 50$
$A\log_e(9) = 50$
$A = \dfrac{50}{\log_e 9}$

b. (i) Find y when $t=15$.

$$y = \left(\frac{50}{\log_e 9}\right)\log_e(t+1)$$

$$y = \left(\frac{50}{\log_e 9}\right)\log_e(15+1) = \left(\frac{50}{\log_e 9}\right)\log_e(16) \text{ units}$$

(ii) $\left(\dfrac{50}{\log_e 9}\right)\log_e(t+1) = \dfrac{1}{\log_e 9}$

$\log_e(t+1) = \dfrac{1}{50}$

$e^{1/50} = t+1$

$t = e^{1/50} - 1\ \mu\text{sec}$

c. (i) $y = \left(\dfrac{50}{\log_e 9}\right)\log_e(t+1)$

$t = \left(\dfrac{50}{\log_e 9}\right)\log_e(y+1)$

$\dfrac{t\log_e 9}{50} = \log_e(y+1)$

$e^{\frac{t}{50}\log_e 9} = y+1$

$y = f^{-1}(t) = e^{\frac{t}{50}\log_e 9} - 1 = e^{\log_e 9^{t/50}} - 1$

$y = f^{-1}(t) = 9^{t/50} - 1$

Alternatively:

$\dfrac{t}{50}\log_e 9 = \log_e(y+1)$

$\log_e 9^{t/50} = \log_e(y+1)$

$9^{t/50} = y+1$

$y = f^{-1}(t) = 9^{t/50} - 1$

(ii) Range f^{-1} = Domain $f(t) : (0, \infty)$.

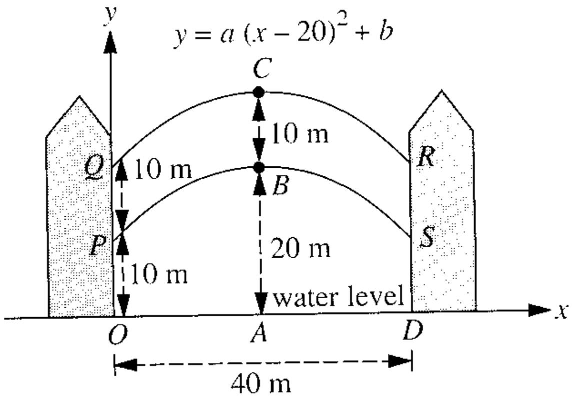
d. Translation of 1 unit in the positive direction, parallel to the t axis.

UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

LOGARITHMIC FUNCTIONS QUESTION 2: BRIDGE

QUESTION 2 (10 Marks)
A bridge is to be built across a 40 metre wide creek. The architect draws on a set of axes a cross-section of the bridge, which is symmetrical about the line AC, as illustrated below.



The water level is along OD. The architect introduces the x and y coordinate system for calculation purposes as shown. For example, point S has coordinates $(40, 10)$.

a. State the coordinates of Q, C and R.

____

____

1 mark

b. If the curve QCR is a parabola with equation $y = a(x-20)^2 + b$, show that $a = -\frac{1}{40}$ and $b = 30$.

____

____

____

____

____

____

____

3 marks

c. (i) State the coordinates of the points P and B.

1 mark

(ii) Hence find the average rate of change in height of the bridge between P and B.

2 marks

d. If curve PB is modelled by the equation $y = k\log_e(x+5)+c$, using algebra, show that $k = \dfrac{10}{\log_e 5}$ and $c = 0$.

3 marks

SOLUTION

QUESTION 2

a $Q = (0, 20)$
$C = (20, 30)$
$R = (40, 20)$

b $y = a(x-20)^2 + b$
$(0, 20) \Rightarrow 20 = 400a + b \quad \ldots \quad (1)$
$(20, 30) \Rightarrow 30 = 0 + b$
$b = 30$
Substitute for b in (1)
$20 = 400a + 30$
$-10 = 400a$
$a = -\frac{1}{40}$
So, $y = -\frac{1}{40}(x-20)^2 + 30$

c. (i) $P = (0, 10)$
$B = (20, 20)$

(ii) Av rate $= \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{20-10}{20-0} = \dfrac{1}{2}$ m/m

d $y = k \log_e(x+5) + c$
$(0, 10) \Rightarrow 10 = k \log_e 5 + c \quad \ldots \quad (2)$
$(20, 20) \Rightarrow 20 = k \log_e 25 + c \quad \ldots \quad (3)$
(3)–(2) gives

$$10 = k(\log_e 25 - \log_e 5)$$
$$= k \log_e\left(\frac{25}{5}\right)$$
$$= k \log_e 5$$
$$k = \frac{10}{\log_e 5}$$

Substitute for k in (2):

$$10 = \frac{10}{\log_e 5} \times \log_e 5 + c$$
$$10 = 10 + c$$
$$\therefore c = 0$$

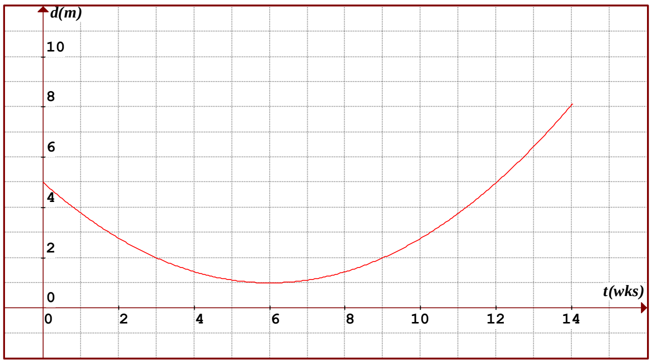
UNIT 1 & 2 MATHEMATICAL METHODS

EXTENDED RESPONSE QUESTION

LOGARITHMIC FUNCTIONS QUESTION 3: FARMER IRENA

QUESTION 3 (15 Marks)

The changes in the depth of the water in Farmer Irena's rain water tank is illustrated below, where d represents the depth, m, of water in the tank, and t represents the time in weeks from January 1st 2006, for $0 \le t \le 14$.



If the depth of water in the tank is modelled by the equation $d(t) = at^2 + bt + c$:

a. (i) Show that $c = 5$.

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1 mark

(ii) By stating two appropriate equations, show that $a = \frac{1}{9}$ and $b = -\frac{4}{3}$.

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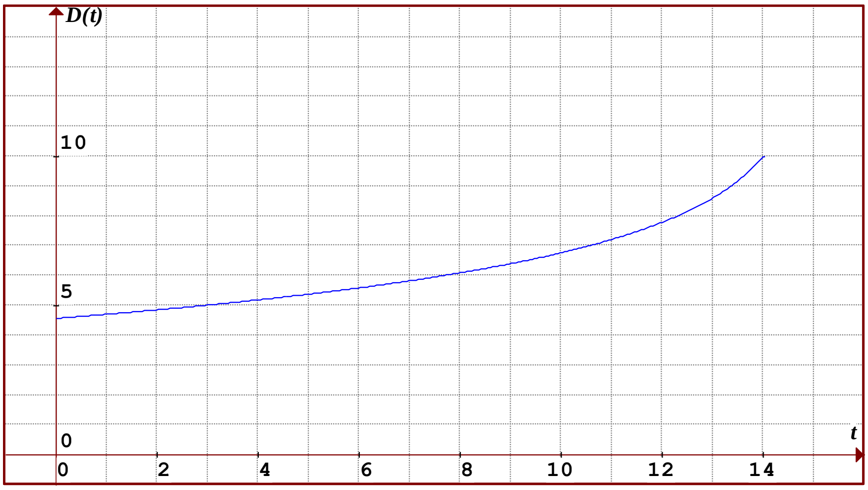
3 marks

b. The depth of water in Farmer Robert's water tank is described by the equation $D(t) = a - b\log_e(15 - t)$ where D represents the depth, m, of water in the tank, and t represents the time in weeks from January 1st 2006, for $0 \le t \le 14$.



(i) If $D(1) = 4.7219$ and $D(4) = 5.2042$ and k_1 and k_2 represent real number values, write an equation in the form $b = \dfrac{k_1}{\log_e(k_2)}$ that could be used to find the value of b.

2 marks

(ii) Hence find the values for a and b. Express your answers to the nearest whole number.

2 marks

(iii) Hence find, correct to 3 decimal places, the time at which the depth of water in Farmer Irena's and Farmer Robert's tank are equal.

2 marks

c. (i) State an expression for the difference in the depths of water in Farmer Irena's and Robert's tanks.

1 mark

(ii) Hence find the exact difference in depths at 3 weeks.

1 mark

(iii) Determine when the difference in the depths of water in the two tanks is at its greatest value. State your answer correct to 3 decimal places.

1 mark

d. When the average monthly rainfall was low (between Weeks 1 and 5), Farmer Irena topped up her water tank with water illegally siphoned from Farmer Robert's supplies.

The depth of water in Farmer Irena's tank across this period of time is modelled by the inverse of $d(t)$. Find the rule describing the inverse function, $d^{-1}(t)$ stating the corresponding domain and range.

3 marks

SOLUTION

QUESTION 3

a. (i) When $t=0$, $d=5$

$\therefore\ 5=c$

(ii) $d = at^2+bt+5$

$(3,2)$: $9a+3b+5=2$

$9a+3b=-3$ — ①

$(9,2)$: $81a+9b+5=2$

$81a+9b=-3$ — ②

② − 3×① :

$$\begin{array}{r} 81a+9b=-3 \\ 27a+9b=-9 \\ \hline 54a=6 \end{array}$$

$a=1/9$ $\therefore\ b=-4/3$

$$\therefore\ d(t)=\frac{t^2}{9}-\frac{4t}{3}+5$$

b. (i) $D = a-b\log_e(15-t)$

(ii) $(1, 4.7219)$: $a-b\log_e 14 = 4.7219$ — ①

$(4, 5.2042)$: $a-b\log_e 11 = 5.2042$ — ②

① − ② : $b\log_e 11 - b\log_e 14 = -0.4823$

$b\log_e\left(\frac{11}{14}\right) = -0.4823$

$$b=\frac{-0.4823}{-0.24116}=2$$

Substitute $b=2$ into ① $\therefore\ a=10$

(iii) Using technology, solve for the point(s) of intersection of the two curves.

Let $\frac{t^2}{9} - \frac{4t}{3} + 5 = 10 - 2\log_e(15-t)$

$t = 0.2898 = 0.290$ weeks

c. (i) Difference in depths $= \left[10 - 2\log_e(15-t)\right] - \left[\frac{t^2}{9} - \frac{4t}{3} + 5\right]$

$= 5 - 2\log_e(15-t) - \frac{t^2}{9} + \frac{4t}{3}$

(ii) Difference = $5 - 2\log_e 12 - \frac{9}{9} + \frac{4(3)}{3} = 8 - 2\log_e 12$ metres.

(iii) Using technology – greatest difference occurs when $t = 7.146$ weeks and the difference at that point is 4.732 metres.

d.

$d(t) = \frac{t^2}{9} - \frac{4t}{3} + 5$

$\therefore\ t = \frac{d^2}{9} - \frac{4d}{3} + 5$

$= \frac{d^2}{9} - \frac{12d}{9} + \frac{45}{9}$

$t = \frac{1}{9}\left(d^2 - 12d + 45\right)$

$= \frac{1}{9}\left[d^2 - 12d + 36 - 36 + 45\right]$

$= \frac{1}{9}\left[(d-6)^2 + 9\right]$

$\therefore\ 9t = (d-6)^2 + 9$

$(d-6)^2 = 9t - 9$

$(d-6) = \sqrt{9t-9}$

$= \sqrt{9(t-1)}$

$= 3\sqrt{t-1}$

$\therefore\ d^{-1} = 3\sqrt{t-1} + 6$

Dom $d^{-1}(t) = \{1 \le t \le 5\}$

Ran $d^{-1}(t) = \{6 \le t \le 12\}$