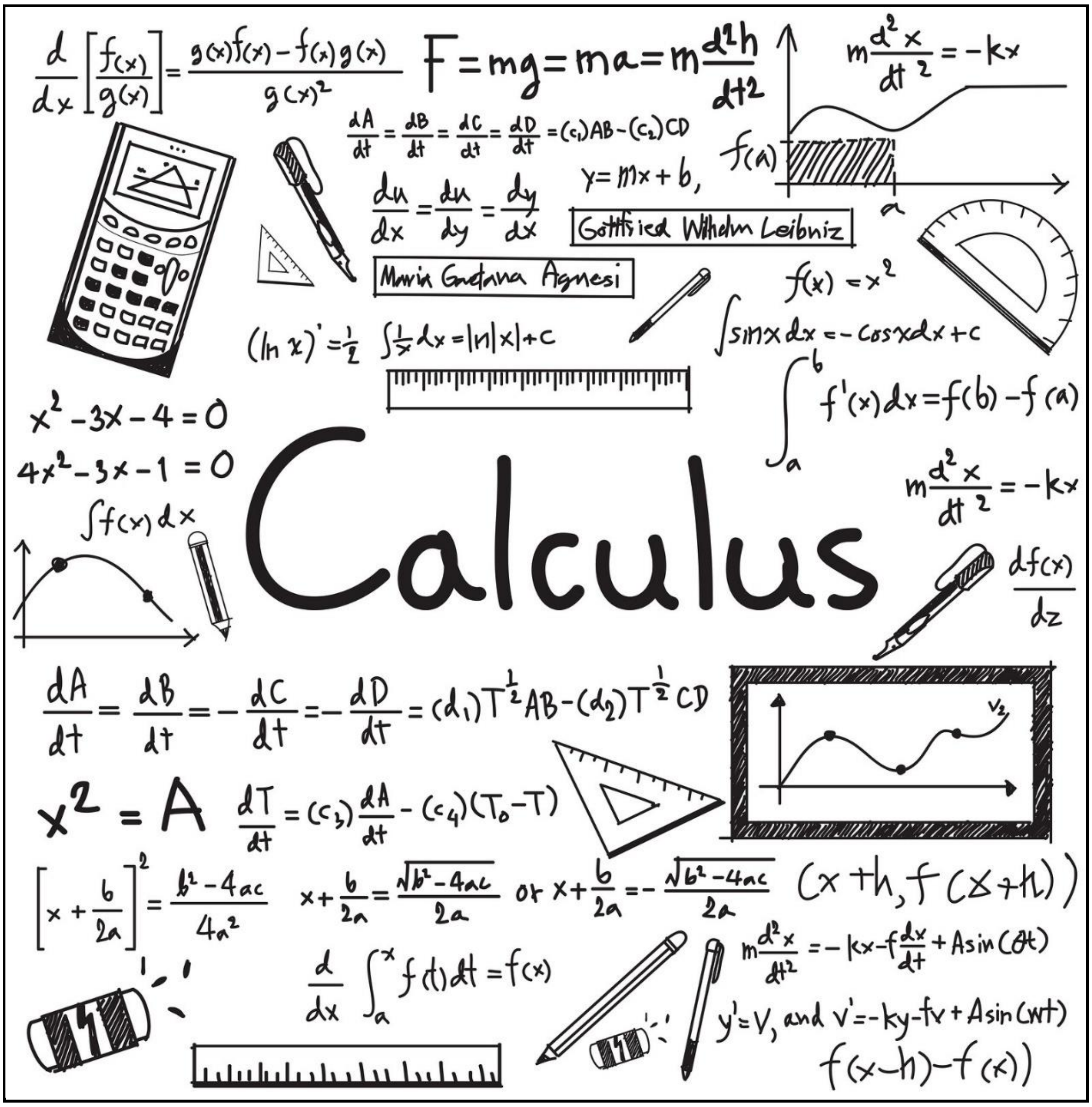


UNIT 3 MATHS METHODS

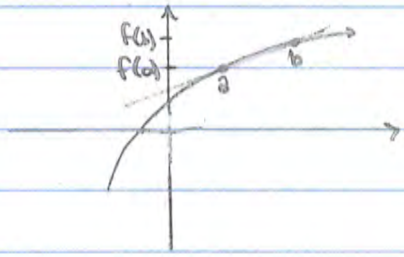
CALCULUS NOTES FOR THE VCAA EXAMS

WRITTEN BY A STUDENT WHO OBTAINED A NEAR PERFECT STUDY SCORE

Calculus

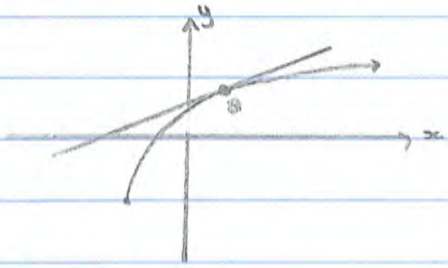
Rates of change:

The average rate of change of a function $f(x)$ between two points $x=a$ and $x=b$ is the gradient of the secant



Using $\frac{\text{rise}}{\text{run}}$ the average rate of change is $\frac{f(b)-f(a)}{b-a}$. It can be vise-versa

The instantaneous rate of change in $f(x)$ at $x=a$ is the gradient of the tangent at $x=a$.



We can either use differentiation by first principle or first derivative to find it.

examples:

1. Calculate the average rate of change of $f(x)=x^2+3x+2$ between $x=1$ and $x=3$.

$f(3)=3^2+3\times3+2=20$

$f(1)=1^2+3\times1+2=6$

$\frac{f(3)-f(1)}{3-1}$

$=\frac{20-6}{3-1}$

$=\frac{14}{2}$

$=7$

2. Calculate the average rate change of $f(x)=x^2+3x+2$ between $x=1$ and $x=(1+h)$. Hence, estimate the gradient of the tangent at $x=1$

$f(1+h)=(1+h)^2+3(1+h)+2$

$=1+2h+h^2+3+3h+2$

$=h^2+5h+6$

$f(1)=1^2+3\times1+2=6$

$\text{Ave}=\frac{(h^2+5h+6)-6}{1+h-1}$

$=\frac{h^2+5h}{h}=h+5$

The gradient of the secant between x and $x=1+h$ is $h+5$. The gradient of the tangent is when $h=0$

$0+5=5$

Differentiation by first principle

For a generalised function $f(x)$ and points $(x, f(x))$ and $(x+h, f(x+h))$ we can generalise the rule for the instantaneous rate of change at any point $(x, f(x))$ by considering what happens as $h \to 0$. The rule for the gradient of the tangent to $y = f(x)$ at any point x is written $f'(x)$ and has rule:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

example:

① Given $f(x) = x^2 + 3x + 2$, find the rule for $f'(x)$ by first principles.

$$f(x+h) = (x+h)^2 + 3(x+h) + 2$$

$$= x^2 + 2xh + h^2 + 3x + 3h + 2$$

$$f'(x) = \lim_{h \to 0} \frac{(x^2 + 2xh + h^2 + 3x + 3h + 2) - (x^2 + 3x + 2)}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2 + 3h}{h}$$

$$= \lim_{h \to 0} 2x + h + 3$$

$$= 2x + 3$$

Differentiation

For any real number n (doesn't have to be rational) the derivative of $f(x) = x^n$ is $f'(x) = n \times x^{n-1}$. When we write a function $y = \ldots$ it is customary to write the derivative as $\frac{dy}{dx}$. For this reason $\frac{d}{dx}$(expression) means the derivative of the expression in the brackets

→ $\frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x)$

→ $\frac{d}{dx}(k \times f(x)) = k \times f'(x)$

→ $\frac{d}{dx}(\text{constant}) = 0$

examples:

① Differentiate $f(x) = x^3 - 2x + 1$

$f'(x) = 3x^2 - 2$

② differentiate $f(x) = \frac{x^2 - 2x}{x}$

$f(x) = 1 - 2x^{-1}$

$f'(x) = 1 + 2x^{-2}$

③ Differentiate $f(x) = x^{5/3}$

$f'(x) = \frac{5}{3} x^{2/3}$

Gradient at a point:

Remember that a tangent is a line drawn on a graph at a point which has the same gradient as the graph at that point. For this reason, the derivative of a function at a point has the same value as the gradient of the tangent at that point.

examples:

① At a point $P(x,y)$ on the graph $f(x) = x^2 - 2x$ a tangent is drawn which makes an angle of $135°$ with the positive x-axis. Find the co-ordinates of P and hence find the equation of tangent.

$m = \tan(135)$

$= -1$

$f'(x) = 2x - 2 = -1$

$2x = 1$

$x = \frac{1}{2}$

when $x = \frac{1}{2}$,

$y = (\frac{1}{2})^2 - 2(\frac{1}{2})$

$= \frac{1}{4} - 1 = \frac{-3}{4}$

$\therefore$ P is $(\frac{1}{2}, \frac{-3}{4})$

tangent line:

$y = -x + c$

when $(\frac{3}{2}, \frac{-3}{4})$,

$\to \frac{-3}{4} = \frac{-1}{2} + c$

$c = \frac{-1}{4}$

$\therefore$ eqn $\to y = -x - \frac{1}{4}$

② A function $f: R\setminus\{0\} \to R$, $f(x) = \frac{a}{x} + bx$. Find the value of a and b given that $f(1) = -1$ and $f'(1) = -5$

$f(x) = ax^{-1} + bx$

$f(1) = a + b = -1$ ①

$f'(x) = -ax^{-2} + b$

$f'(1) = -a + b = -5$ ②

① + ② $\to 2b = -6$

$b = -3$

resub $b = -3$ into ①,

$a - 3 = -1$

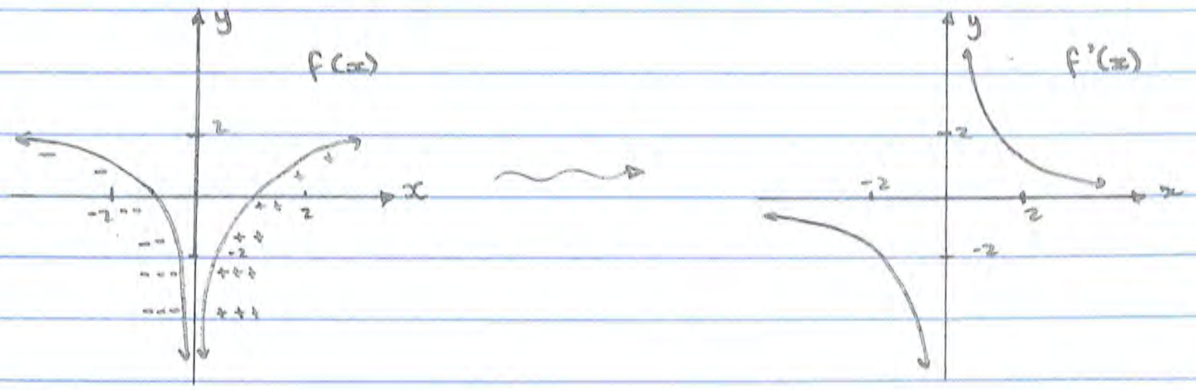
$a = 2$

$\therefore a = 2, b = -3$

Graphing the derivative

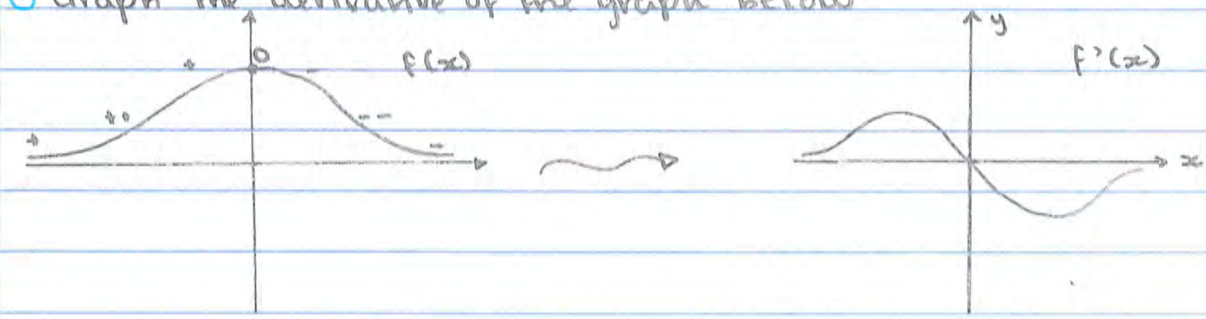
When graphing the derivative some key observations are:

- → when the gradient of $f(x)$ is negative, the graph of $y = f'(x)$ is below the x-axis
- → when the gradient of $f(x)$ is positive, the graph of $y = f'(x)$ is above the x-axis
- → when the gradient of $f(x)$ is zero, the graph of $y = f'(x)$ intersects the x-axis
- → the steeper the gradient, the further the graph of the derivative is from the x-ax



example:

① Graph the derivative of the graph below



Chain rule

If u is a function inside another function y then the chain rule states

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

examples:

① Find $\frac{d}{dt}(2-3t)^9$

$\frac{dy}{dt} = 9(2-3t)^8 \times -3$

$= -27(2-3t)^8$

② $f(x) = \sqrt{2x-5}$. Find $f'(x)$

$f(x) = (2x-5)^{1/2}$

$f'(x) = 1/2\,(2x-5)^{-1/2} \times 2$

$= (2x-5)^{-1/2} = \frac{1}{\sqrt{2x-5}}$

③ Differentiate $(x - \frac{1}{x})^{-1}$

$\frac{dy}{dx}\ -(x - x^{-1})^{-2} \times 1 + x^{-2}$

$= -(1 + \frac{1}{x^2})(x - \frac{1}{x})^{-2}$

④ $f(x) = (x^2 - x)^7$. Find $f'(x)$

$f'(x) = 7(x^2-x)^6 \times 2x-1$

$= 7(2x-1)(x^2-x)^6$

NOTE: $-(x - \frac{1}{x})^{-2}(1 + \frac{1}{x^2}) = -(\frac{x^2}{x} - \frac{1}{x})^{-2}(\frac{x^2}{x^2} + \frac{1}{x^2})$

$= -\left(\frac{x^2-1}{x}\right)^{-2}\left(\frac{x^2+1}{x^2}\right)$

$= -\left(\frac{x}{x^2-1}\right)^2\left(\frac{x^2+1}{x^2}\right)$

$= -\frac{x^2}{(x^2-1)^2} \times \frac{x^2+1}{\cancel{x^2}}$

$= -\frac{(x^2+1)}{(x^2-1)^2}$

∴ multiply choice questions could be right even if your answer is not look like one of the options

Product Rule:

If $f(x) = u(x) \times v(x)$ then $f'(x) = u \times v'(x) + v \times u'(x)$.

This is also commonly written as $\frac{dy}{dx} = u \times \frac{dv}{dx} + v \times \frac{du}{dx}$.

examples:

1. Find $f'(x)$ for $f(x) = x^2(x-2)^3$

let $u = x^2$ $\quad v = (x-2)^3$

$\frac{du}{dx} = 2x$ $\quad \frac{dv}{dx} = 3(x-2)^2$

$$\frac{dy}{dx} = 2x(x-2)^3 + 3x^2(x-2)^2$$
$$= x(x-2)^2(2(x-2) + 3x)$$
$$= x(x-2)^2(2x-4+3x)$$
$$= x(x-2)^2(5x-4)$$

2. Find $\frac{dy}{dx}$ of $y = 2x\sqrt{x-1}$

let $u = 2x$ $\quad v = (x-1)^{1/2}$

$\frac{du}{dx} = 2$ $\quad \frac{dv}{dx} = \frac{1}{2}(x-1)^{-1/2}$

$$\frac{dy}{dx} = 2(x-1)^{1/2} + x(x-1)^{-1/2}$$
$$= (x-1)^{-1/2}(2(x-1) + x)$$
$$= (x-1)^{-1/2}(2x-2+x)$$
$$= (x-1)^{-1/2}(3x-2)$$

The quotient rule:

If $f(x) = \frac{u(x)}{v(x)}$ then $f'(x) = \frac{v \times u'(x) - u \times v'(x)}{[v(x)]^2}$

This is also commonly written as $\frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

It is possible to avoid the quotient rule altogether by a combination of the chain rule and product rule by writing:

$\rightarrow \frac{u(x)}{v(x)} = u(x) \times [v(x)]^{-1}$

example:

1. Differentiate $y = \frac{x+1}{x-1}$

let $u = x+1$ $\quad v = x-1$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{(x-1)-(x+1)}{(x-1)^2}$$
$$= \frac{x-1-x-1}{(x-1)^2}$$
$$= \frac{-2}{(x-1)^2}$$

Alternatively (using the product rule):

$y = (x+1)(x-1)^{-1}$

let $u = x+1$ $\quad v = (x-1)^{-1}$

$\frac{du}{dx} = 1$ $\quad \frac{dv}{dx} = -(x-1)^{-2}$

$$\frac{dy}{dx} = (x-1)^{-1} + (x+1) \times -(x-1)^{-2}$$
$$= \frac{1}{(x-1)} + \frac{-x-1}{(x-1)^2}$$
$$= \frac{x-1}{(x-1)^2} + \frac{-x-1}{(x-1)^2}$$
$$= \frac{-2}{(x-1)^2}$$

limits

A limit is the y-value of a function as x approaches a given value. It exists if and only if the value is the same if you approach it from the left or the right.

$\lim_{x\to a} f(x) = f(a)$ if and only if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$

Consider the function $f(x) = \frac{x^2-1}{x-1}$

$f(1)$ does not exist / not defined

However $f(1.0001) = 2.0001$ and $f(0.9999) = 1.9999$

In fact, as x gets closer and closer to 1 from either above or below, $f(x)$ gets closer to 2. We call this the limit $f(x)$ as x approaches 1 and write $\lim_{x\to 1} f(x) = 2$.

In some cases depending on the side you approach from you have different limits.

Consider the function $f(x) = \frac{1}{x}$:

→ If you approach zero from the negative side, then $f(x) \to -\infty$

→ If you approach from the positive side then $f(x) \to \infty$

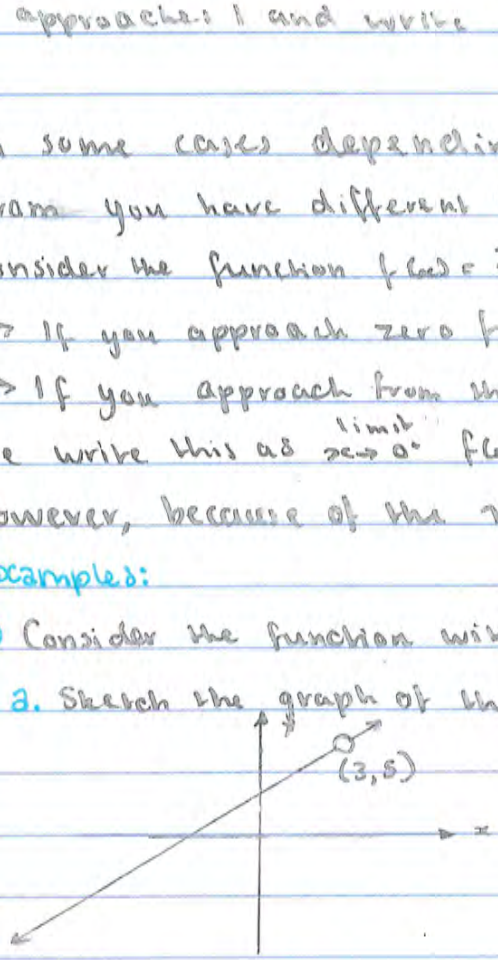
We write this as $\lim_{x\to 0^+} f(x) = \infty$ and $\lim_{x\to 0^-} f(x) = -\infty$

However, because of the rule earlier $\lim_{x\to 0} f(x)$ is undefined.

examples:

① Consider the function with rule $f(x) = \frac{x^2-x-6}{x-3}$

a. Sketch the graph of the function



$y = \frac{(x-3)(x+2)}{x-3}$

$= x+2$ however, $x \neq 3$

y int. → $y=2$

x int. → $x=-2$

b $\lim_{x\to 3} x+2 = 5$

② For the function $f(x) = \begin{cases} x-1, & x<0 \\ 1-x, & x\geq 0 \end{cases}$ calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$

$\lim_{x\to 0^-} x-1 = -1$

$\lim_{x\to 0^+} 1-x = 1$

Continuity

A function is said to be continuous if you can draw the graph without removing your pencil from the page.

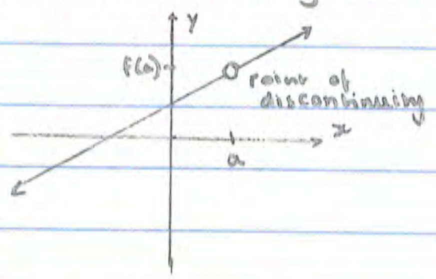
Therefore, a function is said to be continuous at the point $x = a$ if:

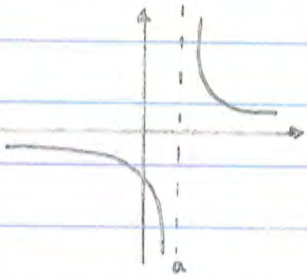
→ $f(a)$ exists

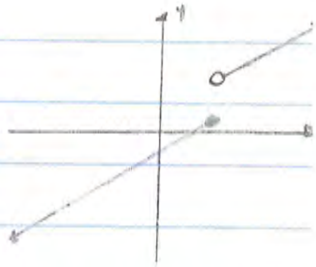
→ As you approach $x = a$ from both directions, the value of $f(x)$ approaches the same value of $f(a)$

∴ At the point $(a, f(a))$, $\lim_{x \to a} f(x) = f(a)$

types of discontinuity:







$f(a)$ doesn't exist
$\lim_{x \to a} f(x) = f(a)$

$f(a)$ doesn't exist
$\lim_{x \to a} f(x) \neq f(a)$

$f(a)$ does exist
$\lim_{x \to a} f(x) \neq f(a)$

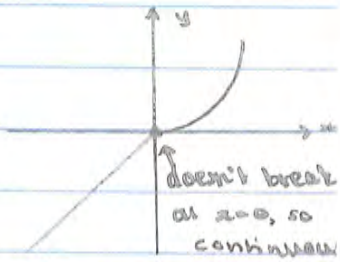
examples:

① Is the graph of $g(x) = \begin{cases} x, & x<0 \\ x^2, & x \geq 0 \end{cases}$, continuous everywhere? If not name the x-values at the point(s) where $g(x)$ is not continuous

$g(0) = 0$ Therefore, $g(x)$ is continuous

$\lim_{x \to 0^-} x = 0$

$\lim_{x \to 0^+} x^2 = 0$



② Determine whether or not the graph of $g(x) = \begin{cases} (x-2)^2+1, & x \leq 2 \\ -2x+1, & x>2 \end{cases}$ is continuous at $x=2$.

$g(2) = (2-2)^2+1$

$= 1$

$\lim_{x \to 2^-} (x-2)^2+1 = 1$

$\lim_{x \to 2^+} -2x+1 = -3$

$1 \neq -3$, so $g(x)$ is not continuous at $x=2$

NOTE:

If the question does not specify to use limits you can use other methods

Conditions of Differentiability:

A derivative exists at a point if the graph is smooth and continuous at that point. You must be able to draw a unique tangent at the point being considered. A function does not have a derivative at:

→ any point of discontinuity

→ at a sharp point

→ any point with a vertical tangent

→ at any end point

We use an open circle at these points, as these points all have multiple or no tangents.

A point on the graph of $y=f(x)$ where the graph is continuous but not differentiable is known as a sharp point.

examples:

① Is the function $f(x) = \begin{cases} x^2, & x \leq 0 \\ x^3, & x > 0 \end{cases}$ differentiable at $x=0$?

$f(0) = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x \to 0} x^2 = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0} x^3 = 0$

$f(x)$ is continuous at $x=0$

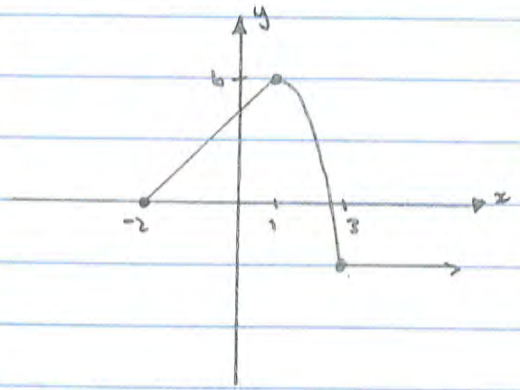
$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0} 2x = 0$

$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0} 3x^2 = 0$

$\therefore f'(x)$ is continuous at $x=0$

$\therefore f(x)$ is a smooth join

② For what values of x is the function differentiable

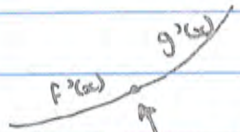


domain → $(-2,1) \cup (1,3) \cup (3,\infty)$

Smooth joins

When functions are joined, one of two scenarios may arise:

The join is smooth



gradients are equal at point of contact i.e. $f'(x) = g'(x)$

The join is not smooth

gradients are not equal at point of contact i.e. $f'(x) \neq g'(x)$

If two curves are joined smoothly:

→ **the curves intersect i.e the x and y values of the individual curves are the same**

→ **the gradients at the point of contact are the same**

example:

1. When $f(x) = 6\sqrt{x} - x - 5$ and $g(x) = x^2$.

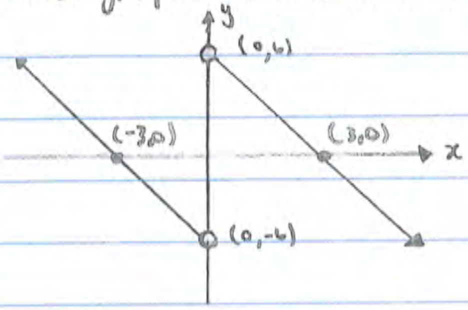
a) Find $f(g(x))$.

$$f(g(x)) = \begin{cases} 6x - x^2 - 5, & x \geq 0 \\ -6x - x^2 - 5, & x < 0 \end{cases}$$

b) let $h(x) = f(g(x))$. Find $h'(x)$

$$h'(x) = \begin{cases} 6 - 2x, & x > 0 \\ -6 - 2x, & x < 0 \end{cases}$$

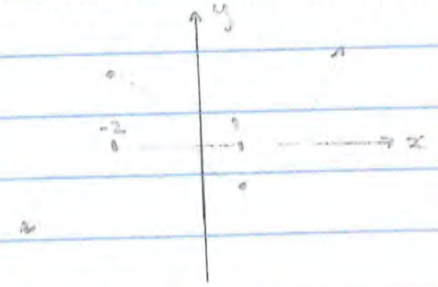
c) Sketch the graph of the derivative function $y = h'(x)$



x int. → $0 = 6 - 2x$
$2x = 6$
$x = 3$
$0 = -6 - 2x$
$2x = -6$
$x = -3$

Strictly increasing and strictly decreasing functions:

A function f is said to be strictly increasing when $a<b$ implies $f(a)<f(b)$ for all a and b in its domain. This definition does not require to be different, or have a non-zero derivative, for all elements of the domain. In other words, end points are included.



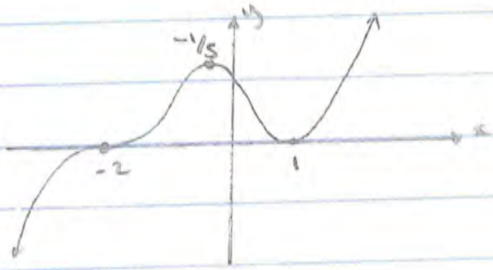
strictly increasing when $x \in (-\infty,-2] \cup [1,\infty)$

strictly decreasing when $x \in [-2,1]$

A function f is said to be strictly increasing when $a<b$ implies $f(a)<f(b)$ for all a and b in its domain. A function is said to be strictly decreasing over an interval when $a<b$ implies $f(a)<f(b)$ for all a and b in its domain.

examples:

For the function $f(x)=(x-1)^2(x+2)^3$ find



a) $\{x : f(x)>0\}$

$0<(x-1)^2(x+2)^3$

$x=-2$ or 1

$\therefore\ x \in (-2,\infty)\setminus\{1\}$

b) $\{x : f'(x)=0\}$

$f'(x) = 2(x-1)(x+2)^3 + (x-1)^2 3(x+2)^2$

$= (x-1)(x+2)^2[2(x+2)+3(x-1)]$

$= (x-1)(x+2)^2(5x+1)$

$f'(x)=0, \quad x=1,-2,-1/5$

c) The value(s) of x which $f(x)$ is strictly increasing

$(-\infty,-1/5] \cup [1,\infty)$

d) The value(s) of x which $f(x)$ is strictly decreasing

$[-1/5, 1]$

e) The value(s) of x for which $f(x)$ changes from an increasing to decreasing function

$x=-1/5$

Tangent Line:

A tangent line to $y=f(x)$ drawn at point $x=a$ will have gradient $m=f'(a)$ and will also pass through point $(a, f(a))$

examples:

① Find equation of tangent line to $y=x^{1/3}$ at the point $(8,2)$.

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$

at $x=8$, $m = \frac{1}{3} \times (8)^{-2/3}$

$= \frac{1}{3} \times \frac{1}{4}$

$= \frac{1}{12}$

$y = \frac{1}{12}x + c$

when $(8,2) \rightarrow 2 = \frac{8}{12} + c$

$\frac{24}{12} = \frac{8}{12} + c$

$c = \frac{16}{12}$

$c = \frac{4}{3}$

$\therefore$ tangent line: $y = \frac{1}{12}x + \frac{4}{3}$

② Find the equation of the tangent to $y = \frac{x+1}{x-1}$ at the point the graph crosses the x-axis.

$0 = \frac{x+1}{x-1} \rightarrow 0 = x+1$

$x = -1$

$\therefore$ point is $(-1, 0)$

let $u = x+1$, $v = x-1$

$\frac{du}{dx} = 1 \quad \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{(x-1)-(x+1)}{(x-1)^2}$

$= \frac{-2}{(x-1)^2}$

at $x=-1$, $m = \frac{-2}{(-1-1)^2}$

$= -\frac{1}{2}$

$y = -\frac{1}{2}x + c$

when $(-1,0) \rightarrow 0 = \frac{1}{2} + c$

$c = -\frac{1}{2}$

$\therefore$ tangent line: $y = -\frac{1}{2}x - \frac{1}{2}$

NOTE:

Remember when writing the answer of the tangent line or normal line you must write: "tangent line: y= ..." or "normal line: y= ..."

Normal Line:

A normal to the graph $y=f(x)$ at $x=a$ is a line which is perpendicular to the tangent. If m_T is the gradient of the tangent m_N is the gradient of the normal then $m_N = \frac{-1}{m_T}$

$m_N \times m_T = -1$

examples:

① Find the equation of the normal

$f(x) = \sqrt{-x}$ @ $x = -1$

when $x = -1$, $y = \sqrt{1} = 1$

point is $(1,1)$

$f(x) = (-x)^{1/2}$

$f'(x) = \frac{1}{2}(-x)^{-1/2} \times -1$

$= \dfrac{-1}{2\sqrt{-x}}$

at $x = -1$, $m_T = -\frac{1}{2}$

$\therefore m_N = 2$

$y = 2x + c$

when $(1,1) \rightarrow 1 = 2 + c$

$c = -1$

$\therefore$ normal: $y = 2x - 1$

② The tangent to $y = (x-2)^2 + 1$ at $x = a$ passes $(0,0)$. Find possible value(s) of a

$y = x^2 - 4x + 5$

$m_T = \dfrac{a^2 - 4a + 5 - 0}{a - 0}$

$= \dfrac{a^2 - 4a + 5}{a}$

$\frac{dy}{dx} = 2x - 4$

when $x = a$, $m_T = 2a - 4$

$\therefore 2a - 4 = \dfrac{a^2 - 4a + 5}{a}$

$2a^2 - 4a = a^2 - 4a + 5$

$a^2 = \pm\sqrt{5}$

NOTE:

This can all easily be done on your calculator by using

tangentLine (expression, variable, value)

or... normalLine (expression, variable, value)

They are under menu, calculus, Tangent/Normal line

Acceleration, Velocity and Displacement:

Velocity and displacement are not quite the same as speed and distance. The difference is they have a direction, therefore they can have a positive or negative sign. Acceleration can also be negative. Symbols for these kinematics are:

- Displacement $\rightarrow x, s$
- Velocity $\rightarrow v$
- Acceleration $\rightarrow a$
- Time $\rightarrow t$

Velocity is measured by the rate of change of the displacement. The average velocity of an object between time t_1 and t_2 is given by the formula:

$$V_{ave} = \frac{x(t_2) - x(t_1)}{t(t_2) - t(t_1)}$$

The average acceleration of an object between times t_1 and t_2 is given by the formula:

$$A_{ave} = \frac{v(t_2) - v(t_1)}{t(t_2) - t(t_1)}$$

The instanteous velocity of an object at time t is then $v = \frac{dx}{dt}$

The instantaneous acceleration of an object at time t is $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

example

① The displacement of an object in meters, t seconds after is begins to move is given by $s(t) = \frac{2}{3}t^3 - \frac{7}{2}t^2 + 3t - 4$. Find the times at which velocity of an object is zero and determine its position and acceleration then

$s(t) = \frac{2}{3}t^3 - \frac{7}{2}t^2 + 3t - 4$

$v(t) = 2t^2 - 7t + 3 = 0$

$t = \frac{1}{2}$ or 3

$a(t) = 4t - 7$

when $t = \frac{1}{2}$,

$a(\frac{1}{2}) = -5\ m/s^2$

$s(\frac{1}{2}) = \frac{-79}{24}\ m$

when $t = 3$,

$a(3) = 5\ m/s^2$

$s(3) = \frac{-17}{2}\ m$

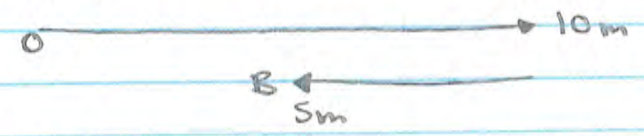
Distance Vs Displacement:

The term displacement (x) describes how far an object has travelled from the origin. It can be positive, zero or negative.

Change in position $= x_{final} - x_{initial} = \Delta x$

Distance (d) describes the length of the path travelled by an object which is always positive or zero.

E.g.



In this case the distance is 15m but the displacement is 7m

Therefore, turning points must be taken into consideration when finding the distance

Note:

Total distance travelled is found by the area under a velocity graph.

example:

① A particle follows the equation for velocity of $v(t) = 12t^2 - 72t + 96$. Find the total distance covered by the particle in the first 6 seconds.

$$0 = 12t^2 - 72t + 96$$
$$0 = t^2 - 6t + 8$$
$$0 = (t-4)(t-2)$$
$$t = 4,\ t = 2$$

Therefore, S.P occurs at $t = 4$ and $t = 2$

$$T.D.T = \int_0^2 12t^2 - 72t + 96\,dt + \int_4^2 12t^2 - 72t + 96\,dt + \int_4^6 12t^2 - 72t + 96\,dt$$

$$= \left[4t^3 - 36t^2 + 96t\right]_0^2 + \left[4t^3 - 36t^2 + 96t\right]_4^2 + \left[4t^3 - 36t^2 + 96t\right]$$

$$= (32 - 144 + 192) + (80 - 256 + 576 - 384) + (864 - 1296 + 576 - 64$$

$$= 80 + 16 + 80$$

$$= 176\text{ m}$$

<u>OR</u>

Determine the equation (displacement) and look at its location and distance travelled between stationary points.

Stationary points:

Consider an object in motion with displacement given by $s(t)$. The derivative, $s'(t) = v(t)$ or velocity. If velocity is zero, the object is said to be stationary. So a stationary point is any point where the derivative of the function is zero.
Therefore, the point $(a, f(a))$ on the graph $y = f(x)$ is a stationary point if and only if $f'(a) = 0$.

examples:

1. Find the co-ordinates of the stationary point with equation $y = \frac{x^2+1}{x}$ when $x > 0$

$u = x^2 + 1 \quad v = x$

$\frac{du}{dx} = 2x \quad \frac{dv}{dx} = 1$

$\frac{dy}{dx} = \frac{2x^2 - x^2 + 1}{x^2}$

$= \frac{x^2 - 1}{x}$

$\Rightarrow 0 = (x-1)(x+1)$

$x = 1$ or $x = -1 \leftarrow$ outside domain

when $x = 1$, $y = \frac{1^2+1}{1} = 2$

$\therefore$ s.p is $(1, 2)$

2. A function has equation $f(x) = x^2 + ax + b$. It has a s.p at $(2, 1)$. Find the values a and b.

$f(2) = 4 + 2a + b = 1$ ①

$f'(x) = 2x + a$

$f'(2) = 4 + a = 0$

$a = -4$

sub $a = -4$ into ①

$4 - 8 + b = 1$

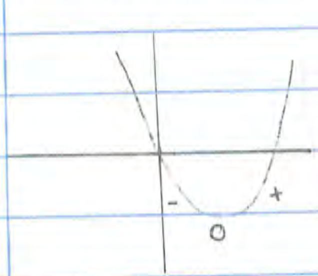
$-4 + b = 1$

$b = 5$

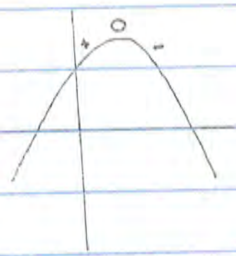
$\therefore a = -4$ and $b = 5$

Classifying stationary points

Stationary points can be classified as either a maximum, minimum or a stationary point of inflection. Stationary point of inflection this occurs when the gradient becomes zero but does not change sign.



maximum



minimum

stationary point of inflection

There are a number of ways to classify these stationary points. If you have a calculator draw it. For polynomial sign diagrams are good and intuition should not be underestimated. However, you can also use the second derivative which is written as $f''(x)$ or $\frac{d^2y}{dx^2}$.

When using the second derivative note that:

If $f'(a)=0$ and $f''(a)>0$ then the point $(a, f(a))$ is always a minimum

If $f'(a)=0$ and $f''(a)<0$ then the point $(a, f(a))$ is always a maximum

If $f'(a)=0$ and $f''(a)=0$ then the point could be anything, including a point of inflection

example:

① A function f has rule $f(x)=\frac{x^2+1}{x}$. The graph of $y=f(x)$ has a stationary point at $(-1,-2)$. What type of point is this?

$f(x) = x + x^{-1}$

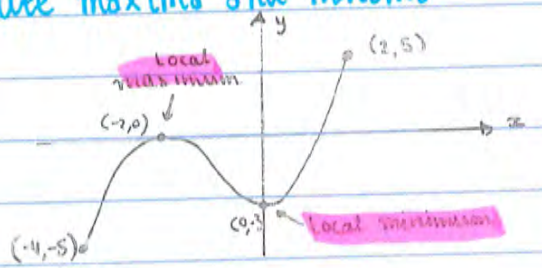
$f'(x) = 1 - x^{-2}$

$f''(x) = 2x^{-3}$

$f''(-1) = \frac{2}{(-1)^3}$

$= -2$ ∴ negative, so $(-1,-2)$ is a maximum

Absolute maxima and minima



Looking at this graph we can see that local maximums and local minimums are different to absolute maximums and absolute minimums

also known as global

example

① A function g is defined by the rule $g: [\frac{-3}{2}, 3] \to \mathbb{R}$, $g(x) = x^3 - 3x - 2$. What is the range?

$g'(x) = 3x^2 - 3 = 0$

$x^2 = 1$

$x = \pm 1$

$g(1) = 1 - 3 - 2 = -4$ $(1, -4)$ s.t

$g(-1) = -1 + 3 - 2 = 0$ $(-1, 0)$ s.t

$g(-\frac{3}{2}) = (-\frac{3}{2})^3 - 3(-\frac{3}{2}) - 2$

$= -\frac{27}{8} + \frac{9}{2} - 2$

$= -\frac{27}{8} + \frac{36}{8} - \frac{16}{8} = \frac{-7}{8}$ $(\frac{-3}{2}, \frac{-7}{8})$ s.t

$g(3) = 27 - 9 - 2 = 16$ $(3, 16)$ s.t

∴ range → $[-4, 16]$

Optimisation

How to solve optimisation problems:

1. Draw a diagram. Even a quick sketch can be very helpful if a diagram is not provide.
2. Remember, you need one dependent variable and one independent variable to use calculus. If there are multiple variables, try and express them in terms of the one independent variable using the available information
3. Check that all your answers make sense in the context of the original problem. For example, are negative answers permitted?
4. Always check end points
5. Answer the whole question - if a question asks for maximum area, give the maximum area, not just the dimensions that give the maximum area

NOTE:

The functions "fMin" and "fMax" are very useful at finding the absolute minimums and absolute maximums of the function. The syntax is fMin (function, variable) and if you want to use a restricted domain it is fMin (function, variable, min, max). Defining the function first is usually very useful.

Derivative of other functions

The derivative of the function e^x is itself. This therefore result in e^x being easy to simplify

$$\Rightarrow \frac{d}{dx}\left(e^{f(x)}\right) = f'(x)e^{f(x)}$$

The derivative of $\log_e(x)$ follows the below rule.

$$\Rightarrow \frac{d}{dx}\left(\log_e(f(x))\right) = \frac{f'(x)}{f(x)}$$

The derivative for circular functions follow the rules below.

$$\left.\begin{aligned} &\Rightarrow \frac{d}{dx}\left(\sin(f(x))\right) = f'(x).\cos(f(x)) \\ &\Rightarrow \frac{d}{dx}\left(-\sin(x)\right) = -\cos(x) \end{aligned}\right\} \sin(x)$$

$$\left.\begin{aligned} &\Rightarrow \frac{d}{dx}\left(\cos(f(x))\right) = -f'(x).\sin(f(x)) \\ &\Rightarrow \frac{d}{dx}\left(-\cos(x)\right) = \sin(x) \end{aligned}\right\} \cos(x)$$

$$\left.\begin{aligned} \Rightarrow \frac{d}{dx}\left(\tan(x)\right) &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x) \end{aligned}\right\} \tan(x)$$

Anti-differentiation

Antidifferentiation is the opposite to the process of differentiation. Integration is the inverse of differentiation

$\Rightarrow \frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ and $\int\left[\frac{d}{dx}F(x)\right]dx = F(x) + c$

Too anti-diff an algebraic expression, we do the following

$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$, $n \neq -1$ also, $\int kx^n\,dx = k\int x^n\,dx$

also, $\int g(x) + f(x)\,dx = \int g(x)\,dx + \int f(x)\,dx$

NOTE:

→ '$+c$' is included when the question says find 'the' antiderivative However, you can add it all the time.

→ When writing out the intergral 'dx' must be put at the end to indicate that the given expression is to be antidifferentiated with respect to x. Thus, this can changing depending on what variable you're using.

example

① Anti-differentiate $f(x) = \frac{3x^2 - 2x}{\sqrt{x}}$

$f(x) = \frac{x^{1/2}(3x^{3/2} - 2x^{1/2})}{x^{1/2}}$

$\int 3x^{3/2} - 2x^{1/2}\,dx = 3\times\frac{2}{5}x^{5/2} - 2\times\frac{2}{3}x^{3/2}$

$= \frac{6}{5}x^{5/2} - \frac{4}{3}x^{3/2} + c$

② $f'(x) = 2x + 2$, if $f(1) = 0$, find $f(x)$

$f(x) = \int 2x + 2\,dx$

$= x^2 + 2x + c$

$f(1) = 1 + 2 + c = 0$

$c = -3$

$\Rightarrow f(x) = x^2 + 2x - 3$

Some other standard anti-derivatives

- $\int (e^{kx+a})\,dx = \frac{1}{k}e^{kx+a} + c$
- $\int (ax+b)^n\,dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$

trig functions:

- $\int \sin(ax+b)\,dx = \frac{-1}{a}\cos(ax+b) + c$
- $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + c$

others:

- $\int \frac{1}{x}\,dx = \log_e(x) + c$
- $k\int \frac{g'(x)}{g(x)}\,dx = k\cdot\log_e(g(x)) + c$

NOTE:

When $x < 0$, $\int \frac{1}{x}\,dx = \log_e(-x) + c$ because $\frac{d}{dx}(\log_e(-x)) = \frac{1}{x}$

Antidifferentiating hybrids

If the equation of the curve changes across the required interval - you must write separate integrals for each section of the curve/equation.

Given $f(x) = \begin{cases} x^2 - 5x - 6, & x \geq -1 \\ 4, & x < -1 \end{cases}$

$$\int f(x)\,dx = \begin{cases} \frac{x^3}{3} - \frac{5x^2}{2} - 6x + c_1, & x \geq -1 \\ 4x + c_2, & x < -1 \end{cases}$$

NOTE:

Unlike derivating a hybrid you don't have to take into consideration whether the equality signs change.

Definite integrals

If $f(x)$ is a continuous function between $x=a$ and $x=b$ then the definite integral of $f(x)$ between $x=a$ and $x=b$ is written:

$$\int_a^b f(x)\,dx$$

a and b are terminals

integrand

measure

anti-derivative

Theorem: $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(b)$

Special properties:

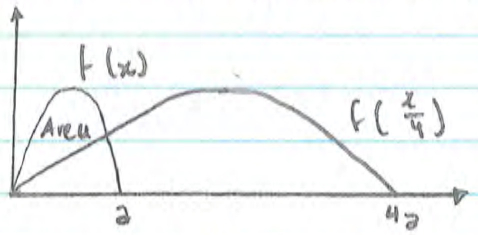
$\int_a^a f(x)\,dx = 0$

$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

example:

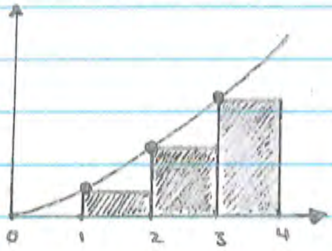
1 If $\int_0^2 f(x)\,dx = 3$, then $\int_0^8 f(\frac{1}{4}x) + 2\,dx$ is?



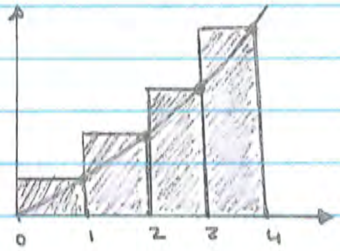
Area 4×

$$\int_0^8 f(\tfrac{x}{4}) + 2\,dx = 4 \times \int_0^2 f(x)\,dx + \int_0^8 2\,dx$$
$$= 4 \times 3 + [2x]_0^8$$
$$= 12 + 16$$
$$= 28 \text{ units}^2$$

Area under a curve



Left end-point endpoint estimate



Right end-point endpoint estimat

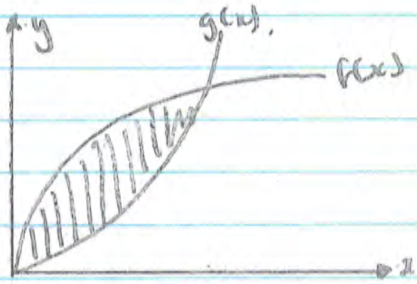
One way to make an estimate of the area under a curve is to calculate the area off the estimated rectangles and add them all together.

To find the exact area under a graph you use a definite integral. If the graph goes under the graph a negative number will be produced so turn it into a positive. If the graph crosses the x-axis break it into smaller parts.

Area between two graphs

To find the area between two graphs use the following formulae:

$$\int_b^a f(x) - g(x)\, dx$$

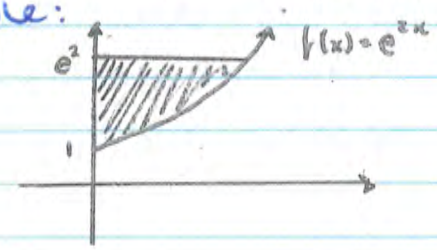


Finding the area using an inverse:

To find an area on a graph sometimes with the information provided its a lot easier to find the inverse to find the area. You must like usual swap the x and y values.

Example:

1



Find the area of the shaded region.

$f^{-1}(x) \rightarrow x = e^{2y}$

$2y = \log_e(x)$

$y = \frac{1}{2}\log_e(x)$

$\Rightarrow \text{Area} = \int_1^{e^2} \frac{1}{2}\log_e(x)\, dx$

$= \frac{e^2+1}{2}$ units ^{2}.

NOTE:

For approximation the smaller the width the more accurate and the average value is best e.g. $\frac{(\text{left end-point} + \text{right end-point areas})}{2}$

Average Value:

The formula when you examining a function $f(x)$ over the domain $[a,b]$ is: $\frac{1}{b-a}\int_a^b f(x)\,dx$

example:

① The average value of $y=(x-k)^2$ over the domain $[-2,2]$ is $\frac{5}{3}$. Determine the value of k.

$\frac{5}{3} = \frac{1}{4}\int_{-2}^{2} (x-k)^2\,dx$

$20/3 = \left[\frac{1}{3}(x-k)^3\right]_{-2}^{2}$

$20/3 = \left(\frac{1}{3}(2-k)^3\right) - \left(\frac{1}{3}(-2-k)^3\right)$

$20 = (2-k)^3 + (2+k)^3$

$20 = (2-k+2+k)(4-4k+k^2-(2-k)(2+k)+4+4k+k^2)$

$= 4(8+2k^2-(4-k^2))$

$5 = 4+3k^2$

$1 = 3k^2$

$k^2 = \frac{1}{3}$

$k = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$

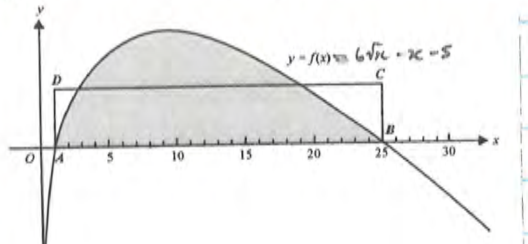
* Always look for short cuts $(a+b)(a^2-ab+b^2)$

NOTE:

The average value can also be found when, the area underneath the graph over a specific domain equals the area with length of the domain. Then the average value is the height of the rectangle.

example:

① Find the height AD such that the area to AB CD equals to the area of the shaded region.



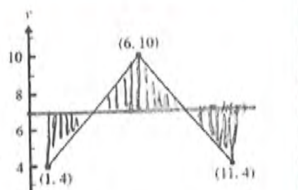
Ave value $= \frac{1}{24}\int_1^{25} 6\sqrt{x} - x - 5\,dx$

$= \frac{1}{24}\left[6\times\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 - 5x\right]_1^{25}$

$= \frac{8}{3}$

$\Rightarrow |AD| = 8/3$

② The average value of the graph below is.



7 as shown above.