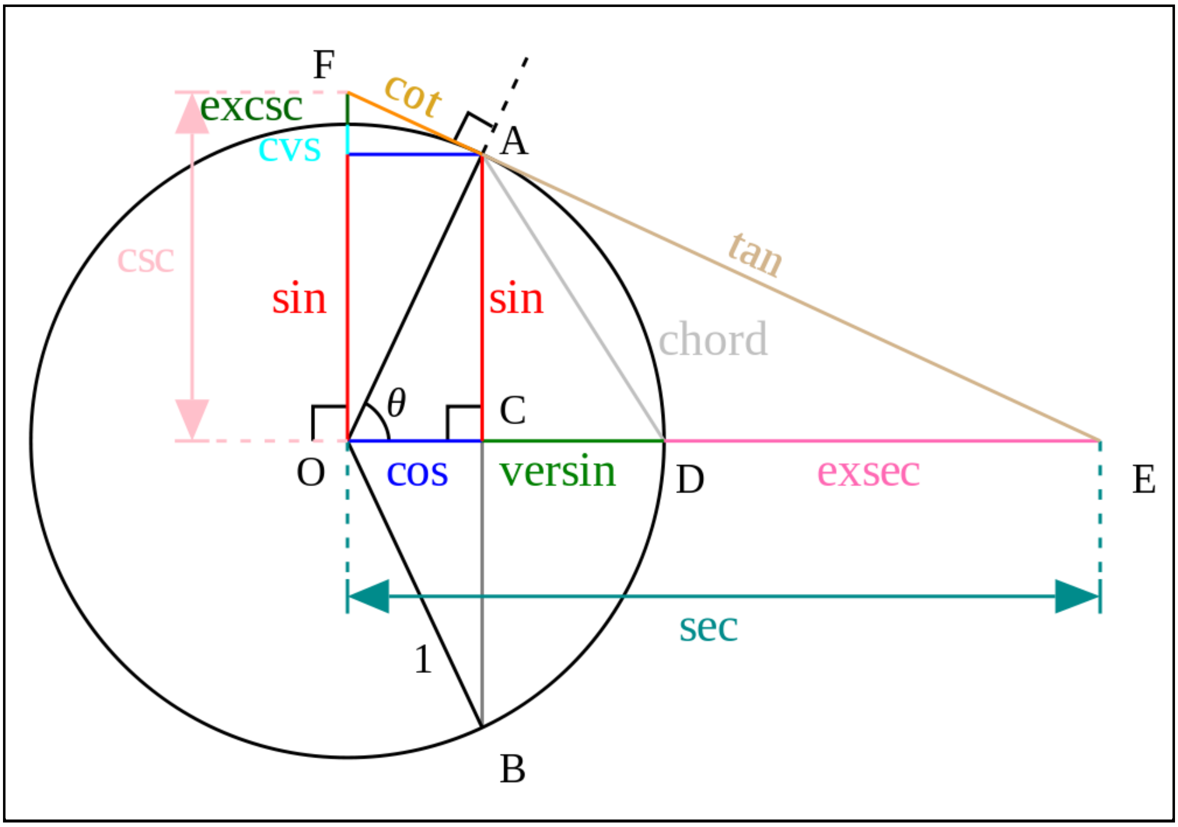

SOLVING TRIGONOMETRIC EQUATIONS

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SOLVING TRIGONOMETRIC EQUATIONS

Step 1: Write all expressions in terms of one trigonometric function.

Step 2: Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation.

Step 3: Use the sign in front of the constant on the right hand side to determine the quadrants in which the solutions are to lie (use **CAST**).

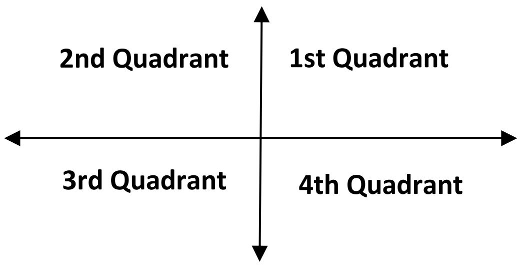
Step 4: Calculate the reference angle i.e. the first quadrant solution. If the exact value cannot be determined:

Press Inverse Sin, Cos or Tan of the number on the right hand side of the equation (but ignore the sign).

For example: $Sin^{-1}(number\ on\ RHS\ of\ equation\ but\ ignore\ the\ sign)$

(Ensure that the calculator is in Radian Mode).

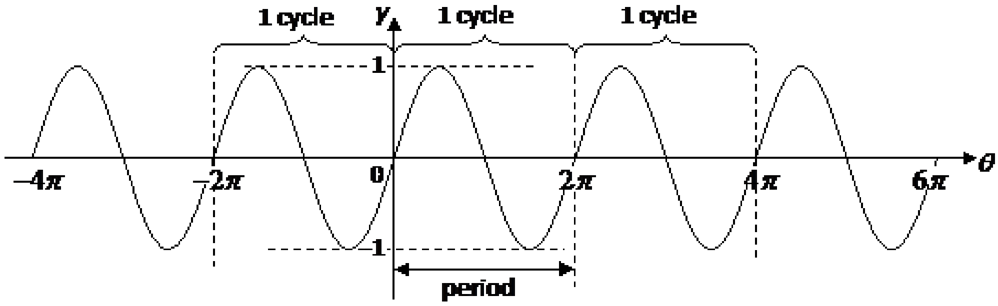
Step 5: Solve for the variable (usually x or θ). Let the angle equal the rule describing angles in the quadrants in which the solutions are to lie.



Note: First Quadrant Angle = FQA

Let angle = FQA if solution lies in 1st Quadrant.
Let angle = $\pi - FQA$ if solution lies in 2nd Quadrant.
Let angle = $\pi + FQA$ if solution lies in 3rd Quadrant.
Let angle = $2\pi - FQA$ if solution lies in 4th Quadrant.

Step 6: Evaluate all possible solutions by observing the given domain. This is done by adding or subtracting the **PERIOD** to each of the solutions, until the angles fall outside the given domain.



For sine and cosine functions: $Period = \dfrac{2\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

For tangent functions: $Period = \dfrac{\pi}{|The\ number\ in\ front\ of\ the\ variable|}$

Always look closely at the brackets in the given domain and consider whether the upper and lower limits can be included in your solutions.

DO NOT discard any solution until the final step.

Step 7: Eliminate solutions that do not lie within the specified domain.

Note: Students may solve trigonometric equations by rearranging the domain.

QUESTION 1

Solve the following equation for x: $2\sin x = \sqrt{3}$, $x \in [0, 4\pi]$.

Solution

Transpose the given equation so that the trigonometric expression (and the angle) is on one side of the equation, and the constants are located on the other side of the equation:

$$\sin x = \frac{\sqrt{3}}{2}$$

Calculate the reference angle (the first quadrant solution):

$$x = Sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$$

Use the sign in front of the constant on the right hand side to determine the quadrants in which the solutions are to lie:

Solutions are to lie in the quadrants where sine is positive i.e. the 1st and 2nd quadrants:

S✓	A✓
T	C

Solve for the variable (usually x). Let the angle equal the rule describing angles in the quadrants in which the solutions are to lie:

$x = \frac{\pi}{3}$ and $x = \pi - 1st\ Quadrant\ Angle$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$\therefore \left\{x : x = \frac{\pi}{3}, \frac{2\pi}{3}\right\}$$

Evaluate all possible solutions by adding and/or subtracting the PERIOD to each of the calculated answers observing the given domain:

$$T = \frac{2\pi}{1} = 2\pi = \frac{6\pi}{3}$$

$$\left\{x : x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}\right\}$$

SOLVING SIMPLE TRIGONOMETRIC EQUATIONS
WORKSHEET 1

QUESTION 1
Solve each equation over the interval $(-8, 8)$. State your answers correct to 4 decimal places.

(a) $\tan\theta = 4$

(b) $\cos\theta = 0.84$

(c) $\sin\theta = 0.63$

SOLUTIONS

QUESTION 1

(a)

$\tan\theta = 4$

$\theta = \tan^{-1}(4) = 1.32582$

$$\begin{array}{c|c} S & A \\ \hline T & C \end{array}$$

Solutions to lie in Quadrants 1 + 3

$\theta = 1.32582, \ \pi + 1.32582$

$\theta = 1.32582, \ 4.46741$

Domain: $(-8, 8)$ $\therefore$ Add / subtract period $= \frac{\pi}{1} = \pi$

$\theta = -4.95737, -1.81577, 1.32582, 4.46741, 7.609$

$\therefore \theta = -4.9574, -1.8158, 1.3258, 4.4674, 7.6090$

(b)

$\cos\theta = 0.84$

$$\begin{array}{c|c} S & A \\ \hline T & C \end{array}$$

$\cos^{-1}(0.84) = 0.573513$

Solutions to lie in Quadrants 1 + 4

$\theta = 0.573513, \ 2\pi - 0.573513$

$= 0.573513, \ 5.70967$

Domain: $(-8, 8)$ $\therefore$ Add / subtract period $= \frac{2\pi}{1} = 2\pi$

$$\theta = \begin{matrix} 6.8567 \\ 0.573513 \\ -5.70967 \end{matrix}, \quad \begin{matrix} 5.70967 \\ -0.573513 \\ -6.8567 \end{matrix}$$

$\therefore \theta = -6.8567, -5.7097, -0.5735, 0.5735, 5.7097, 6.8567$

(c)

$\sin\theta = 0.63$

$\sin^{-1}(0.63) = 0.681553$

Solutions to lie in Quadrants 1+2

S	A
T	C

$\theta = 0.681553, \ \pi - 0.681553$

$= 0.681553, \ 2.46004$

Domain: $(-8, 8)$ $\therefore$ Add / subtract period $= \frac{2\pi}{1} = 2\pi$

6.96474	~~8.74323~~
0.681553	2.46004
−5.60163	−4.12315

$\therefore \theta = -5.6016, -4.1232, 0.6816, 2.4600, 6.9647$

SOLVING SIMPLE TRIGONOMETRIC EQUATIONS
WORKSHEET 2

QUESTION 1

Solve each equation over the interval $[0, 2\pi)$.

(a) $2\cos x + 4 = 5$

(b) $2\sin x - 1 = 0$

(c) $5\cos x+\sqrt{3}=3\cos x$

(d) $6\tan x+\sqrt{3}=3\tan x$

QUESTION 2

Solve each equation over the interval $[0, 2\pi]$.

(a) $5\cos 2x = 3\cos 2x - 1$

(b) $\sin\left(\frac{x}{2}\right) = \frac{\sqrt{2}}{2}$

(c) $\sin x + \cos x = 0$

(d) $2\cos(3x) + \sqrt{3} = 0$

SOLUTIONS

QUESTION 1

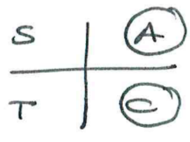
(a)

$$2\cos x + 4 = 5$$

$$2\cos x = 1$$

$$\cos x = \frac{1}{2}$$

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$



$$\therefore\ x = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$$

$$x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

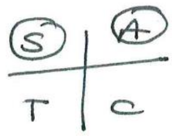
(b)

$$2\sin x - 1 = 0$$

$$2\sin x = 1$$

$$\sin x = \frac{1}{2}$$

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$



$$x = \frac{\pi}{6},\ \pi - \frac{\pi}{6}$$

$$x = \frac{\pi}{6},\ \frac{5\pi}{6}$$

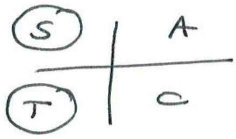
(c)

$$5\cos x + \sqrt{3} = 3\cos x$$

$$2\cos x = -\sqrt{3}$$

$$\cos x = -\frac{\sqrt{3}}{2}$$

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$



$$x = \pi - \frac{\pi}{6},\ \pi + \frac{\pi}{6}$$

$$x = \frac{5\pi}{6},\ \frac{7\pi}{6}$$

(d)

$6\tan x + \sqrt{3} = 3\tan x$

$6\tan x - 3\tan x = -\sqrt{3}$

$3\tan x = -\sqrt{3}$

$\tan x = \frac{-\sqrt{3}}{3}$

$\tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \frac{\pi}{6}$

S	A
T	C

solutions to lie in Quadrants 2 + 3

$x = \pi - \frac{\pi}{6}, \quad 2\pi - \frac{\pi}{6}$

$x = \frac{5\pi}{6}, \quad \frac{11\pi}{6}$

QUESTION 2

(a)

$5\cos 2x = 3\cos 2x - 1$

$5\cos 2x - 3\cos 2x = -1$

$2\cos 2x = -1$

$\cos 2x = -\frac{1}{2}$

S	A
T	C

$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

solutions to lie in Quadrants 2 + 4

$2x = \pi - \frac{\pi}{3}, \quad \pi + \frac{\pi}{3}$

$2x = \frac{2\pi}{3}, \quad \frac{4\pi}{3}$

$\therefore x = \frac{2\pi}{6}, \quad \frac{4\pi}{6}$

Domain: $[0, 2\pi]$ $\therefore$ Add/subtract period $= \frac{2\pi}{2} = \pi = \frac{6\pi}{6}$

$x = \frac{2\pi}{6}, \frac{4\pi}{6}, \frac{8\pi}{6}, \frac{10\pi}{6}$

$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

(b)

$$\sin\left(\frac{x}{2}\right) = \frac{\sqrt{2}}{2}$$

$$\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$$

S	A
T	C

Solutions to lie in Quadrants 1 + 2

$$\frac{x}{2} = \frac{\pi}{4}, \ \pi - \frac{\pi}{4}$$

$$\frac{x}{2} = \frac{\pi}{4}, \ \frac{3\pi}{4}$$

$$\therefore x = \frac{2\pi}{4}, \ \frac{6\pi}{4}$$

$$\therefore x = \frac{\pi}{2}, \ \frac{3\pi}{2}$$

(c)

$$\sin x + \cos x = 0$$

$$\sin x = -\cos x$$

$$\frac{\sin x}{\cos x} = \frac{-\cos x}{\cos x}$$

$$\tan x = -1$$

$$\tan^{-1}(1) = \frac{\pi}{4}$$

Solutions to lie in Quadrants 2 + 4

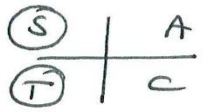
$$x = \pi - \frac{\pi}{4}, \ 2\pi - \frac{\pi}{4}$$

$$x = \frac{3\pi}{4}, \ \frac{7\pi}{4}$$

(d)

$$2\cos 3x + \sqrt{3} = 0$$

$$\cos 3x = \frac{-\sqrt{3}}{2}$$



$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$$

Solutions to lie in Quadrants 2 + 3

$$3x = \pi - \frac{\pi}{6}, \quad \pi + \frac{\pi}{6}$$

$$3x = \frac{5\pi}{6}, \quad \frac{7\pi}{6}$$

$$x = \frac{5\pi}{18}, \quad \frac{7\pi}{18}$$

Domain : $[0, 2\pi]$ $\therefore$ Add/subtract period $= \frac{2\pi}{3} = \frac{12\pi}{18}$

$$x = \frac{5\pi}{18}, \frac{7\pi}{18}, \frac{17\pi}{18}, \frac{19\pi}{18}, \frac{29\pi}{18}, \frac{31\pi}{18}$$

SOLVING SIMPLE TRIGONOMETRIC EQUATIONS
WORKSHEET 3

QUESTION 1
Solve each equation over the given domain.

(a) $\sqrt{3} + \tan 2x = 0$, $[-2\pi, 2\pi)$

(b) $\cos(\pi x) = 0.5$, $[-1, 2)$

(c) $\sin\left(\frac{x}{2}\right)-1=0$, $[0, 8\pi)$

(d) $\tan\left(x+\frac{\pi}{3}\right)=1$, $[0, 2\pi]$

QUESTION 2
Solve each equation over the given domain.

(a) $\tan(3x+2)=\sqrt{3}$, $(-\pi, \pi)$

(b) $-2\cos\left[\pi(x-1)\right]-1=0$, (-3, 3)

(c) $-5\sin\left(3x+\pi\right)=0$, [-2π, 0]

(d) $2\cos\left(x+\frac{\pi}{4}\right)-\sqrt{3}=0$, $[0, 2\pi]$

(e) $-\frac{2}{\sqrt{3}}\sin\left(x+\frac{\pi}{3}\right)-1=0$, $[-2\pi, 2\pi]$

SOLUTIONS

QUESTION 1

(a)

$\sqrt{3} + \tan 2x = 0$

$\tan 2x = -\sqrt{3}$

$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$

S	A
T	C

Solutions to lie in Quadrants $2 + 4$

$2x = \pi - \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}$

$2x = \frac{2\pi}{3}, \quad \frac{5\pi}{3}$

$x = \frac{2\pi}{6}, \quad \frac{5\pi}{6}$

Domain: $[-2\pi, 2\pi)$ $\therefore$ Add/subtract period $= \frac{\pi}{2} = \frac{3\pi}{6}$

$x = \frac{-10\pi}{6}, \frac{-7\pi}{6}, \frac{-4\pi}{6}, \frac{-\pi}{6}, \frac{2\pi}{6}, \frac{5\pi}{6}, \frac{8\pi}{6}, \frac{11\pi}{6}$

$x = -\frac{5\pi}{3}, -\frac{7\pi}{6}, \frac{-2\pi}{3}, -\frac{\pi}{6}, \frac{\pi}{3}, \frac{5\pi}{6}, \frac{4\pi}{3}, \frac{11\pi}{6}$

(b)

$\cos(\pi x) = 0.5, \quad [-1, 2)$

$\cos^{-1}(0.5) = \frac{\pi}{3}$

S	A
T	C

Solutions to lie in Quadrants $1 + 4$

$\pi x = \frac{\pi}{3}, \quad 2\pi - \frac{\pi}{3}$

$\pi x = \frac{\pi}{3}, \quad \frac{5\pi}{3}$

$x = \frac{1}{3}, \quad \frac{5}{3}$

Domain: $[-1, 2)$ $\therefore$ Add/subtract period $= \frac{2\pi}{2} = 2 = \frac{6}{3}$

$x = -\frac{1}{3}, \frac{1}{3}, \frac{5}{3}$

(c)

$$\sin\left(\frac{x}{2}\right) - 1 = 0$$

$$\sin\left(\frac{x}{2}\right) = 1$$

As $\sin\left(\frac{\pi}{2}\right) = 1$:

$$\sin\left(\frac{x}{2}\right) = \sin\left(\frac{\pi}{2}\right)$$

$$\therefore \frac{x}{2} = \frac{\pi}{2}$$

$$\therefore x = \pi$$

Domain: $[0, 8\pi)$ $\therefore$ Add/subtract period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$

$$\therefore x = \pi, 5\pi$$

(d)

$$\tan\left(x + \frac{\pi}{3}\right) = 1$$

S	A
T	C

$$\tan^{-1}(1) = \frac{\pi}{4}$$

Solutions to lie in Quadrants 1 + 3

$$\therefore x + \frac{\pi}{3} = \frac{\pi}{4}, \quad \pi + \frac{\pi}{4}$$

$$x + \frac{\pi}{3} = \frac{\pi}{4}, \quad \frac{5\pi}{4}$$

$$x + \frac{4\pi}{12} = \frac{3\pi}{12}, \quad \frac{15\pi}{12}$$

$$x = -\frac{\pi}{12}, \quad \frac{11\pi}{12}$$

Domain: $[0, 2\pi]$ $\therefore$ Add/subtract period $= \frac{\pi}{1} = \frac{12\pi}{12}$

$$x = -\frac{\pi}{12}, \quad \frac{11\pi}{12}, \quad \frac{23\pi}{12}$$

$$\therefore x = \frac{11\pi}{12}, \quad \frac{23\pi}{12}$$

QUESTION 2

(a)

$$\tan(3x+2) = \sqrt{3}$$

$$\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$$

S	A
T	C

Solutions to lie in Quadrants 1+3

$$3x+2 = \frac{\pi}{3}, \quad \pi + \frac{\pi}{3}$$

$$3x+2 = \frac{\pi}{3}, \quad \frac{4\pi}{3}$$

$$3x = \frac{\pi}{3} - 2, \quad \frac{4\pi}{3} - 2$$

$$3x = \frac{\pi - 6}{3}, \quad \frac{4\pi - 6}{3}$$

$$x = \frac{\pi - 6}{9}, \quad \frac{4\pi - 6}{9}$$

Domain: $(-\pi, \pi)$ $\therefore$ Add/subtract period $= \frac{\pi}{3} = 1.0472$

~~3.8712~~
2.824
1.7768
-0.317601 0.729597
-1.3648
-2.412
~~-3.4592~~

$$\therefore x = \frac{\pi-6}{9} - \frac{3\pi}{9} - \frac{3\pi}{9}, \quad \frac{\pi-6}{9} - \frac{3\pi}{9}, \quad \frac{\pi-6}{9}$$

$$\frac{4\pi-6}{9}, \quad \frac{4\pi-6}{9} + \frac{3\pi}{9}, \quad \frac{4\pi-6}{9} + \frac{3\pi}{9} + \frac{3\pi}{9}$$

$$x = \frac{\pi-6-6\pi}{9}, \quad \frac{\pi-6-3\pi}{9}, \quad \frac{\pi-6}{9}, \quad \frac{4\pi-6}{9}, \quad \frac{4\pi-6+3\pi}{9},$$

$$\frac{4\pi-6+6\pi}{9}$$

$$x = \frac{-7\pi-6}{9}, \quad \frac{-2\pi-6}{9}, \quad \frac{\pi-6}{9}, \quad \frac{4\pi-6}{9}, \quad \frac{7\pi-6}{9}, \quad \frac{10\pi-6}{9}$$

(b)

$$-2\cos\left[\pi(x-1)\right]-1=0$$

$$\cos\left[\pi(x-1)\right]=-\frac{1}{2}$$

$$\cos^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{3}$$

S	A
T	C

Solutions to lie in Quadrants 2 + 3

$$\pi(x-1)=\pi-\frac{\pi}{3},\ \pi+\frac{\pi}{3}$$

$$\pi(x-1)=\frac{2\pi}{3},\ \frac{4\pi}{3}$$

$$x-1=\frac{2}{3},\ \frac{4}{3}$$

$$x=\frac{5}{3},\ \frac{7}{3}$$

Domain: $(-3,3)$ $\therefore$ Add/subtract period $=\frac{2\pi}{\pi}=2=\frac{6}{3}$

$$x=-\frac{7}{3},\ -\frac{5}{3},\ -\frac{1}{3},\ \frac{1}{3},\ \frac{5}{3},\ \frac{7}{3}$$

(c)

$$-5\sin(3x+\pi)=0$$

$$\sin(3x+\pi)=0$$

As $\sin 0 = 0$

$$\sin(3x+\pi)=\sin(0)$$

$$3x+\pi=0$$

$$3x=-\pi$$

$$x=-\pi/3$$

Domain: $[-2\pi, 0]$ $\therefore$ Add/subtract period $=\frac{2\pi}{3}$

$$x=-\frac{5\pi}{3},\ \frac{3\pi}{3},\ \frac{-\pi}{3}$$

(d)

$$2\cos\left(x+\frac{\pi}{4}\right)-\sqrt{3}=0$$

$$\cos\left(x+\frac{\pi}{4}\right)=\frac{\sqrt{3}}{2}$$

S	A
T	C

$$\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$$

Solutions to lie in Quadrants 1 + 4

$$x+\frac{\pi}{4}=\frac{\pi}{6},\ 2\pi-\frac{\pi}{6}$$

$$x+\frac{\pi}{4}=\frac{\pi}{6},\ \frac{11\pi}{6}$$

$$x+\frac{3\pi}{12}=\frac{2\pi}{12},\ \frac{22\pi}{12}$$

$$x=-\frac{\pi}{12},\ \frac{19\pi}{12}$$

Domain: $[0, 2\pi]$ $\therefore$ Add/subtract period $=\frac{2\pi}{1}=\frac{24\pi}{12}$

$$x=\cancel{-\frac{\pi}{12}},\ \frac{19\pi}{12},\ \frac{23\pi}{12}$$

$$x=\frac{19\pi}{12},\ \frac{23\pi}{12}$$

(e)

$$-\frac{2}{\sqrt{3}}\sin\left(x+\frac{\pi}{3}\right)-1=0$$

$$\sin\left(x+\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$$

S	A
T	C

Solutions to lie in Quadrants 3 + 4

$$x+\frac{\pi}{3}=\pi+\frac{\pi}{3},\quad 2\pi-\frac{\pi}{3}$$

$$x+\frac{\pi}{3}=\frac{4\pi}{3},\quad \frac{5\pi}{3}$$

$$x=\frac{3\pi}{3},\quad \frac{4\pi}{3}$$

Domain: $[-2\pi, 2\pi]$ $\therefore$ Add/subtract period $=\frac{2\pi}{1}=\frac{6\pi}{3}$

$$x=-\frac{3\pi}{3},\quad -\frac{2\pi}{3},\quad \frac{3\pi}{3},\quad \frac{4\pi}{3}$$

$$x=-\pi,\quad -\frac{2\pi}{3},\quad \pi,\quad \frac{4\pi}{3}$$

GENERAL SOLUTIONS

If the equation of a circular function has no domain or an infinite domain, there will be an infinite number of solutions.

The solutions to such equations are called **general solutions** and are calculated by applying the rules below:

In General:

If $\cos x = a$ then $x = 2n\pi \pm \cos^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

If $\tan x = a$ then $x = n\pi + \tan^{-1}(a)$ where $n \in Z$ and $a \in R$.

If $\sin x = a$ then $x = 2n\pi + \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

or $x = (2n+1)\pi - \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

Alternative Notation:

If $\sin x = a$ then $x = n\pi + (-1)^n \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

Note:

- To find the inverse trigonometric value, substitute the number and sign on the right hand side of the equation after the equation has been transposed.

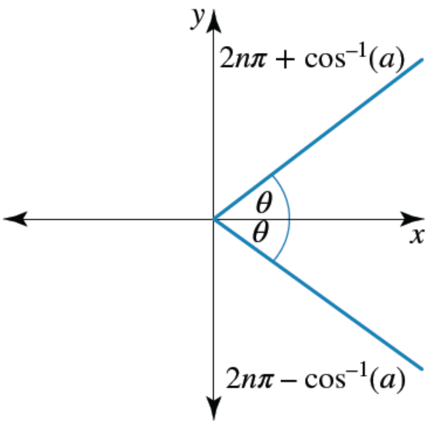
  For example: If $\cos x = -0.5$ then $\cos^{-1}(a) = \cos^{-1}(-0.5) = -\dfrac{\pi}{3}$

- Sometimes, the answer given on the CAS will not look the same as what is obtained when equations are solved algebraically. The answers, however, should be equivalent.

GENERAL SOLUTIONS FOR COSINE

In General: If $\cos x = a$ then $x = 2n\pi \pm \cos^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

To find $\cos^{-1}(a)$ substitute the number and sign on the right hand side of the equation after the equation has been transposed (a). Eg. If $\cos x = -0.5$ then $\cos^{-1}(a) = \cos^{-1}(-0.5) = \frac{2\pi}{3}$



QUESTION 1

Find the general solution for $\cos x = 0.5$.

Solution

If $\cos x = a$ then $x = 2n\pi \pm \cos^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

$$\cos^{-1}(0.5) = \frac{\pi}{3}$$

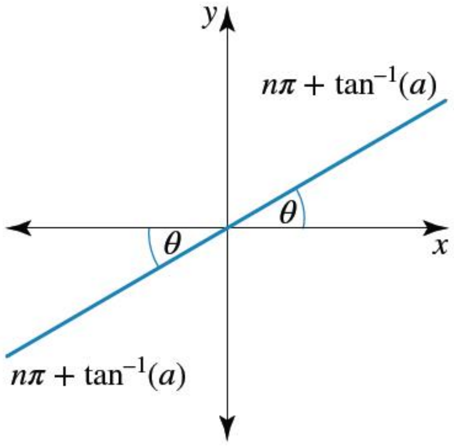
$$\therefore\ x = 2n\pi \pm \frac{\pi}{3} = \frac{6n\pi \pm \pi}{3} = \frac{(6n \pm 1)\pi}{3},\ n \in Z$$

GENERAL SOLUTIONS FOR TANGENT

In General: If $\tan x = a$ then $x = n\pi + \tan^{-1}(a)$ where $n \in Z$ and $a \in R$.

To find $\tan^{-1}(a)$ substitute the number and sign on the right hand side of the equation after the equation has been transposed (a).

For example: If $\tan x = -2$ then $\tan^{-1}(a) = \tan^{-1}(-2) = -1.1071$



QUESTION 2
Find the general solution for $\tan x = 1$.

Solution

If $\tan x = a$ then $x = n\pi + \tan^{-1}(a)$ where $n \in Z$ and $a \in R$.

$\tan^{-1}(1) = \dfrac{\pi}{4}$

$\therefore\ x = n\pi + \dfrac{\pi}{4} = \dfrac{4n\pi + \pi}{4} = \dfrac{(4n+1)\pi}{4},\ n \in Z$

GENERAL SOLUTIONS FOR SINE

If $\sin x = a$ then $x = 2n\pi + \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

or $x = (2n+1)\pi - \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

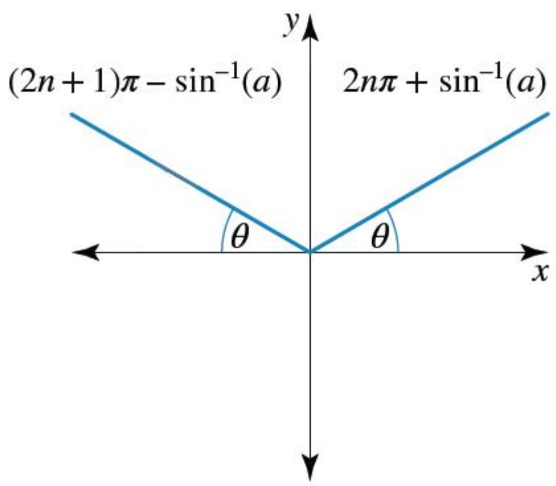
Alternative Notation:

If $\sin x = a$ then $x = n\pi + (-1)^n \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

To find $\sin^{-1}(a)$ substitute the number and sign on the right hand side of the equation after the equation has been transposed (a).

For example: If $\sin x = -0.5$ then $\sin^{-1}(a) = \sin^{-1}(-0.5) = -\frac{\pi}{6}$.

If the solutions for a are positive:



If a is negative:

Use Quadrant 4 rules to calculate the basic angle. This means that angles will be negative.

QUESTION 3
Find the general solution for $\sin x = 0.5$.

Solution

If $\sin x = a$ then $x = 2n\pi + \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

or $x = (2n+1)\pi - \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

$x = 2n\pi + \sin^{-1}(a)$ or $x = (2n+1)\pi - \sin^{-1}(a)$

$\sin^{-1}(0.5) = \dfrac{\pi}{6}$

$x = 2n\pi + \dfrac{\pi}{6}$ or $x = (2n+1)\pi - \dfrac{\pi}{6}$

$x = \dfrac{12n\pi + \pi}{6}$ or $x = \dfrac{6(2n+1)\pi - \pi}{6}$

$x = \dfrac{(12n+1)\pi}{6}$ or $x = \dfrac{(12n\pi + 6\pi - \pi)}{6} = \dfrac{(12n\pi + 5\pi)}{6} = \dfrac{(12n+5)\pi}{6}$

Alternatively:

If $\sin x = a$ then $x = n\pi + (-1)^n \sin^{-1}(a)$ where $n \in Z$ and $-1 \le a \le 1$.

Therefore:

$x = n\pi + (-1)^n \sin^{-1}(a)$

$x = n\pi + (-1)^n \dfrac{\pi}{6}$

Check the answer to this question on the CAS calculator. Are these two sets of answers equivalent?

GENERAL SOLUTIONS OF TRIGONOMETRIC EQUATIONS
WORKSHEET 1

COSINE EQUATIONS

QUESTION 1
Find $2\cos x = 1$.

Solution

QUESTION 2
Find $2\cos\left(3x + \frac{\pi}{6}\right) = 1$.

Solution

QUESTION 3

Find $2\cos\left(2x+\frac{\pi}{3}\right)=1$.

Solution

QUESTION 4

Find $3\cos\left(x-4\right)=1$.

Solution

TANGENT EQUATIONS

QUESTION 5
Find $\tan x = 1$.

Solution

QUESTION 6
Find $\tan\left(\frac{\pi}{3} - x\right) = 0$.

Solution

QUESTION 7
Find $3\tan(2x - 5) + 4 = 0$.

Solution

SINE EQUATIONS – NOTATION 1

QUESTION 8

Find $2\sin x - \sqrt{3} = 0$.

Solution

QUESTION 9

Find $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right) = 1$.

Solution

QUESTION 10
Find $\sin(2x+3)=-1$.

Solution

SINE EQUATIONS – NOTATION 2

QUESTION 11
Find $2\sin x-\sqrt{3}=0$.

Solution

QUESTION 12

Find $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$.

Solution

QUESTION 13

Find $\sin(2x+3)=-1$.

Solution

WORKSHEET 1 – SOLUTIONS

COSINE EQUATIONS

If $\cos(\theta) = (a)$

represents entire angle

represents number and sign on RHS when equation is transposed to isolate cos and its angle

then $(\theta) = 2n\pi \pm \cos^{-1}((a))$

If $\cos\theta = a$ then $\theta = 2n\pi \pm \cos^{-1}(a)$

If $\cos(ENTIRE\ ANGLE) = RHS$ then $ENTIRE\ ANGLE = 2n\pi \pm \cos^{-1}(RHS)$

QUESTION 1

Find $2\cos x = 1$.

Solution

$$\text{Entire Angle} = 2n\pi \pm \cos^{-1}(RHS)$$

$$\cos x = \frac{1}{2}$$

$$x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$

$$x = 2n\pi \pm \frac{\pi}{3} \quad n \in Z$$

QUESTION 2

Find $2\cos\left(3x+\frac{\pi}{6}\right)=1$.

Solution

$$2\cos\left(3x+\frac{\pi}{6}\right)=1$$

$$\cos\left(3x+\frac{\pi}{6}\right)=\frac{1}{2}$$

$$\text{Entire Angle} = 2n\pi \pm \cos^{-1}(\text{RHS})$$

$$\downarrow$$

$$3x+\frac{\pi}{6} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$$

$$3x+\frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

$$3x = 2n\pi \pm \frac{2\pi}{6} - \frac{\pi}{6}$$

$$3x = 2n\pi + \frac{\pi}{6},\ 2n\pi - \frac{3\pi}{6}$$

$$3x = \frac{12n\pi+\pi}{6},\ \frac{12n\pi-3\pi}{6}$$

$$x = \frac{12n\pi+\pi}{18},\ \frac{12n\pi-3\pi}{18}$$

$$x = \frac{(12n+1)\pi}{18},\ \frac{(4n-1)\pi}{6} \quad n \in Z$$

QUESTION 3

Find $2\cos\left(2x+\frac{\pi}{3}\right)=1$.

Solution

$$2\cos\left(2x+\frac{\pi}{3}\right)=1$$

$$\cos\left(2x+\frac{\pi}{3}\right)=\frac{1}{2}$$

If $\cos\left(2x+\frac{\pi}{3}\right)=\frac{1}{2}$ then $2x+\frac{\pi}{3} = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$

$$2x+\frac{\pi}{3} = 2n\pi \pm \frac{\pi}{3}$$

$$2x = 2n\pi \pm \frac{\pi}{3} - \frac{\pi}{3}$$

$$2x = 2n\pi,\ 2n\pi - \frac{2\pi}{3}$$

$$x = n\pi,\ n\pi - \frac{\pi}{3} \quad n \in Z$$

QUESTION 4

Find $3\cos(x-4)=1$.

Solution

$3\cos(x-4)-2=0$

$\cos(x-4)=\frac{2}{3}$

Entire Angle $= 2n\pi \pm \cos^{-1}(RHS)$

$\cos(x-4)=\frac{2}{3}$ then $(x-4)=2n\pi \pm \cos^{-1}\left(\frac{2}{3}\right)$

$x = 2n\pi \pm \cos^{-1}\left(\frac{2}{3}\right)+4$

TANGENT EQUATIONS

If $\tan(\theta) = (a)$

represents entire angle

represents number and sign on RHS when equation is transposed to isolate tan and its angle

then $(\theta) = n\pi + \tan^{-1}(a)$

If $\tan\theta = a$ then $\theta = n\pi + \tan^{-1}(a)$

If $\tan(ENTIRE\ ANGLE) = RHS$ then $ENTIRE\ ANGLE = n\pi + \tan^{-1}(RHS)$

QUESTION 5

Find $\tan x = 1$.

Solution

$\tan x = 1$ then $x = n\pi + \tan^{-1}(1)$

$x = n\pi + \frac{\pi}{4}, \quad n \in Z$

QUESTION 6

Find $\tan\left(\frac{\pi}{3} - x\right) = 0$.

Solution

$\tan\left(\frac{\pi}{3} - x\right) = 0$

Entire Angle $= n\pi + \tan^{-1}(\text{RHS})$

$\frac{\pi}{3} - x = n\pi + \tan^{-1}(0)$

$-x = n\pi - \frac{\pi}{3}$

$x = \frac{\pi}{3} - n\pi$

$x = \left(\frac{1}{3} - n\right)\pi \quad , \; n \in Z$

QUESTION 7

Find $3\tan(2x - 5) + 4 = 0$.

Solution

$3\tan(2x-5) + 4 = 0$

$\tan(2x-5) = -\frac{4}{3}$

$\tan(2x-5) = -\frac{4}{3}$ then $2x - 5 = n\pi + \tan^{-1}\left(-\frac{4}{3}\right)$

$2x = n\pi + \tan^{-1}\left(-\frac{4}{3}\right) + 5$

$x = \frac{n\pi}{2} + \frac{1}{2}\tan^{-1}\left(-\frac{4}{3}\right) + \frac{5}{2}$

SINE EQUATIONS – NOTATION 1

If $\sin(\theta) = (a)$

represents entire angle

represents number and sign on RHS when equation is transposed to isolate sin and its angle

Then $(\theta) = n\pi + (-1)^n \sin^{-1}((a))$

If $\sin\theta = a$ then $\theta = n\pi + (-1)^n \sin^{-1}(a)$

If $\sin(ENTIRE\ ANGLE) = RHS$ then $ENTIRE\ ANGLE = n\pi + (-1)^n \sin^{-1}(RHS)$

QUESTION 8

Find $2\sin x - \sqrt{3} = 0$.

Solution

$2\sin x - \sqrt{3} = 0$

$\sin x = \frac{\sqrt{3}}{2}$

If $\sin\theta = a$ then $\theta = n\pi + (-1)^n \sin^{-1}(a)$

ENTIRE ANGLE $= n\pi + (-1)^n \sin^{-1}(RHS)$

$\sin x = \frac{\sqrt{3}}{2}$ then $x = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$x = n\pi + (-1)^n \cdot \frac{\pi}{3}$

QUESTION 9
Find $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$.

Solution

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$$

$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$\therefore \quad \text{ENTIRE ANGLE} = n\pi+(-1)^n\sin^{-1}(\text{RHS})$$

$$x+\frac{\pi}{4} = n\pi+(-1)^n\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$x+\frac{\pi}{4} = n\pi+(-1)^n\frac{\pi}{4}$$

$$x = n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}, \quad n\in Z$$

QUESTION 10
Find $\sin(2x+3)=-1$.

Solution

$$\sin(2x+3)=-1$$

$$\therefore 2x+3 = n\pi+(-1)^n\sin^{-1}(-1)$$

$$= n\pi+(-1)^n\left(-\frac{\pi}{2}\right)$$

$$= n\pi-(-1)^n\frac{\pi}{2}$$

$$2x = n\pi-(-1)^n\frac{\pi}{2}-3$$

$$x = \frac{n\pi}{2}-(-1)^n\frac{\pi}{4}-\frac{3}{2}, \quad n\in Z$$

SINE EQUATIONS – NOTATION 2

If $\sin(\theta) = (a)$ then

represents entire angle

represents number and sign on RHS when equation is transposed to isolate sin and its angle

$\theta = 2n\pi + \sin^{-1}(a)$

OR

$\theta = (2n+1)\pi - \sin^{-1}(a)$

If $\sin\theta = a$ then $\theta = 2n\pi + \sin^{-1}(a)$ or $\theta = (2n+1)\pi - \sin^{-1}(a)$

If $\sin(ENTIRE\ ANGLE) = RHS$ then $ENTIRE\ ANGLE = 2n\pi + \sin^{-1}(RHS)$
or $ENTIRE\ ANGLE = (2n+1)\pi - \sin^{-1}(RHS)$

QUESTION 11

Find $2\sin x - \sqrt{3} = 0$.

Solution

$$2\sin x - \sqrt{3} = 0$$

$$\sin x = \frac{\sqrt{3}}{2}$$

$$x = 2n\pi + \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) \text{ OR } (2n+1)\pi - \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$x = 2n\pi + \frac{\pi}{3},\ (2n+1)\pi - \frac{\pi}{3},\ n \in Z$$

QUESTION 12

Find $\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$.

Solution

$$\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)=1$$

$$\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

$$x+\frac{\pi}{4}=2n\pi+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \text{ OR } (2n+1)\pi-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$=2n\pi+\frac{\pi}{4}, \ (2n+1)\pi-\frac{\pi}{4}$$

$$x=2n\pi, \ (2n+1)\pi-\frac{\pi}{2} \qquad n\in Z$$

QUESTION 13

Find $\sin(2x+3)=-1$.

Solution

$$\sin(2x+3)=-1$$

$$\therefore\ 2x+3=2n\pi+\sin^{-1}(-1) \text{ OR } (2n+1)\pi-\sin^{-1}(-1)$$

$$=2n\pi-\frac{\pi}{2}, \ (2n+1)\pi+\frac{\pi}{2}$$

$$2x=2n\pi-\frac{\pi}{2}-3, \ (2n+1)\pi+\frac{\pi}{2}-3$$

$$x=n\pi-\frac{\pi}{4}-\frac{3}{2}, \ (2n+1)\frac{\pi}{2}+\frac{\pi}{4}-\frac{3}{2}, \ n\in Z$$