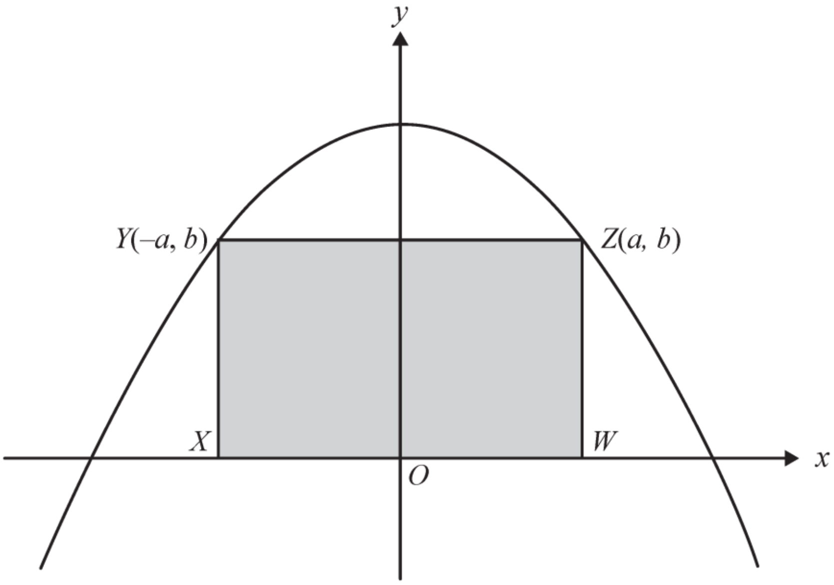
UNIT 3 & 4 MATHEMATICAL METHODS
VCAA EXAMINATION 1 – 2006 TO 2017

DIFFERENTIATION – OPTIMISATION

QUESTION 1 – 2006 (AVERAGE – HARD)

A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram below. The coordinates of Z are (a, b) where a and b are positive real numbers.



a. Find the area, A, of rectangle $XYZW$ in terms of a.

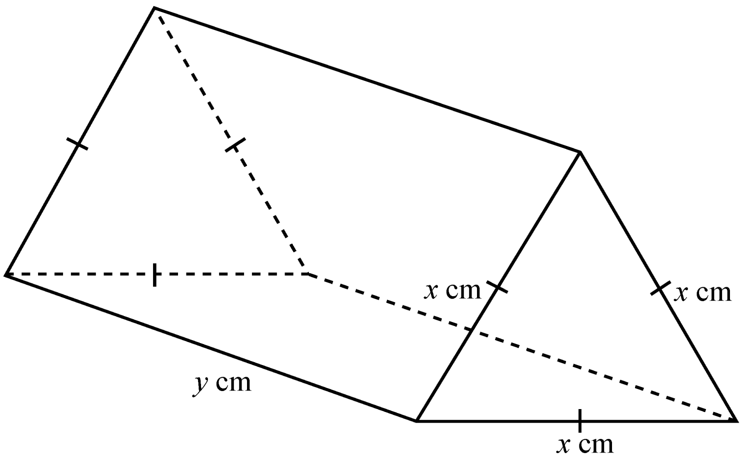
1 mark

b. Find the maximum value of A and the value of a for which this occurs.

3 marks

QUESTION 2 – 2008 (HARD – HARD – HARD)

A plastic brick is made in the shape of a right triangular prism. The triangular end is an equilateral triangle with side length x cm and the length of the brick is y cm.



The volume of the brick is 1000 cm^3.

a. Find an expression for y in terms of x.

b. Show that the total surface area, A cm^2, of the brick is given by

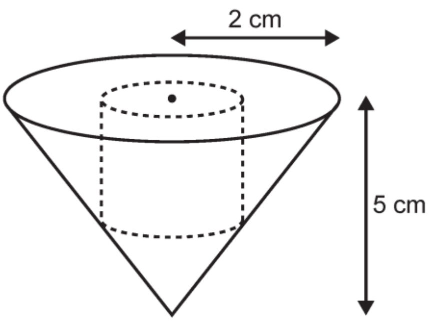
$$A = \frac{4000\sqrt{3}}{x} + \frac{\sqrt{3}x^2}{2}$$

c. Find the value of x for which the brick has minimum total surface area. (You do not have to find this minimum.)

2 + 2 + 3 = 7 marks

QUESTION 3 – 2010 (HARD – HARD – HARD)

A cylinder fits exactly in a right circular cone so that the base of the cone and one end of the cylinder are in the same plane as shown in the diagram below. The height of the cone is 5 cm and the radius of the cone is 2 cm. The radius of the cylinder is r cm and the height of the cylinder is h cm.



For the cylinder inscribed in the cone as shown above

a. find h in terms of r

2 marks

The total surface area, S cm^2, of a cylinder of height h cm and radius r cm is given by the formula

$$S = 2\pi rh + 2\pi r^2.$$

b. find S in terms of r

1 mark

c. find the value of r for which S is a maximum.

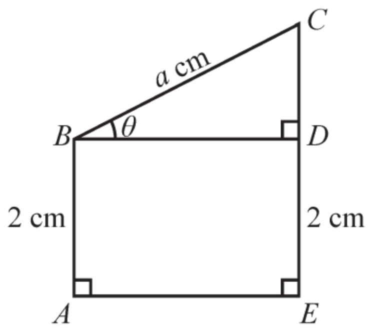
2 marks

QUESTION 4 – 2011 (AVERAGE – AVERAGE – HARD – HARD)

The figure shown represents a wire frame where $ABCE$ is a convex quadrilateral. The point D is on line segment EC with $AB = ED = 2$ cm and $BC = a$ cm, where a is a positive constant.

$$\angle BAE = \angle CEA = \frac{\pi}{2}$$

Let $\angle CBD = \theta$ where $0 < \theta < \frac{\pi}{2}$.



a. Find BD and CD in terms of a and θ.

2 marks

b. Find the length, L cm, of the wire in the frame, including length BD, in terms of a and θ.

1 mark

c. Find $\frac{dL}{d\theta}$, and **hence** show that $\frac{dL}{d\theta} = 0$ when $BD = 2CD$.

2 marks

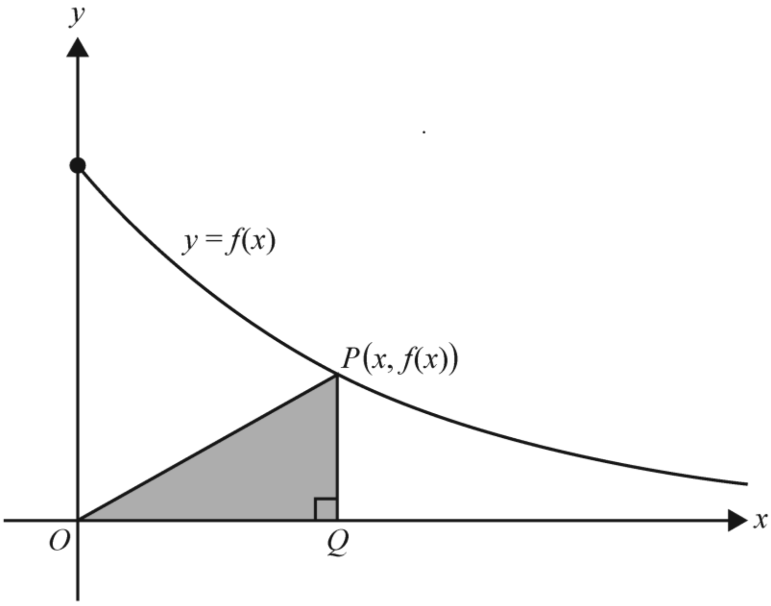
d. Find the maximum value of L if $a = 3\sqrt{5}$.

1 mark

QUESTION 5 – 2013 (AVERAGE – HARD – HARD)

Let $f: [0, \infty) \to R,\ f(x) = 2e^{-\frac{x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.

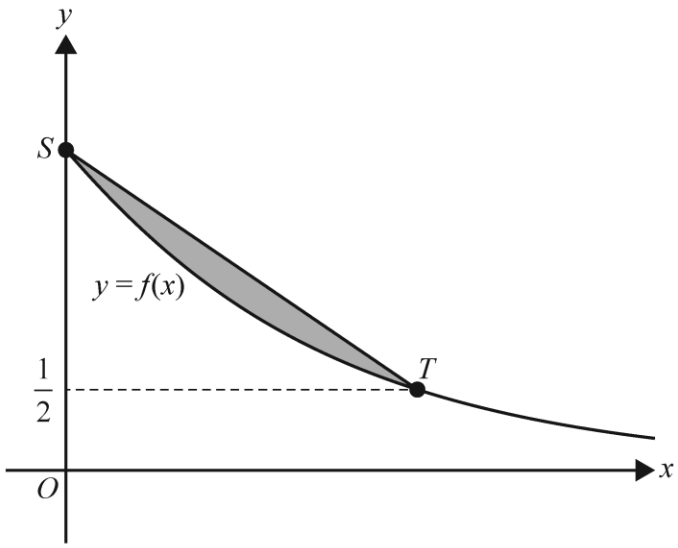


a. Find the area, A, of the triangle OQP in terms of x. 1 mark

b. Find the maximum area of triangle OQP and the value of x for which the maximum occurs. 3 marks

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

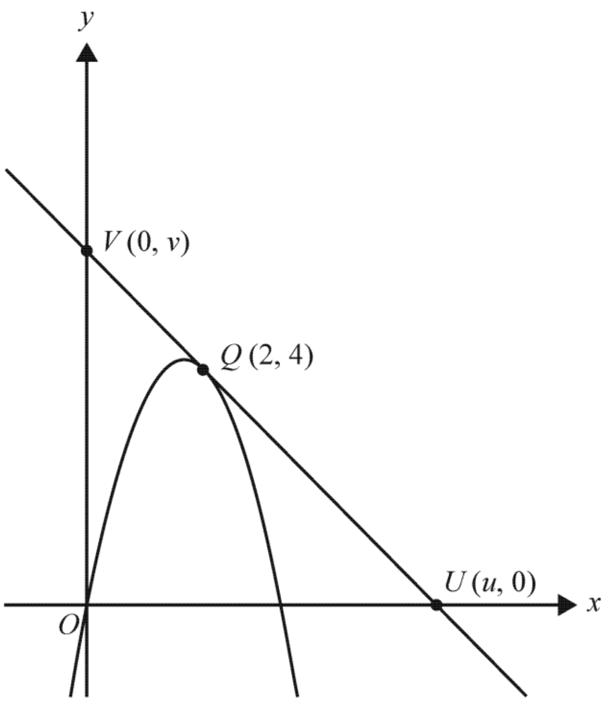
Find the area of the region bounded by the graph of f and the line segment ST. 3 marks



QUESTION 6 – 2014 (HARD – HARD – HARD – HARD)

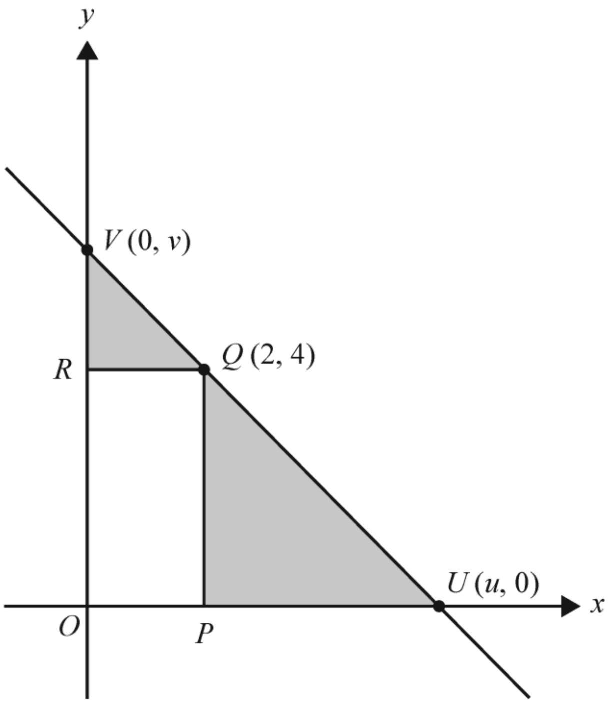
A line intersects the coordinate axes at the points U and V with coordinates $(u, 0)$ and $(0, v)$, respectively, where u and v are positive real numbers and $\frac{5}{2} \le u \le 6$.

a. When $u = 6$, the line is a tangent to the graph of $y = ax^2 + bx$ at the point Q with coordinates $(2, 4)$, as shown.



If a and b are non-zero real numbers, find the values of a and b. 3 marks

b. The rectangle $OPQR$ has a vertex at Q on the line. The coordinates of Q are (2, 4), as shown.



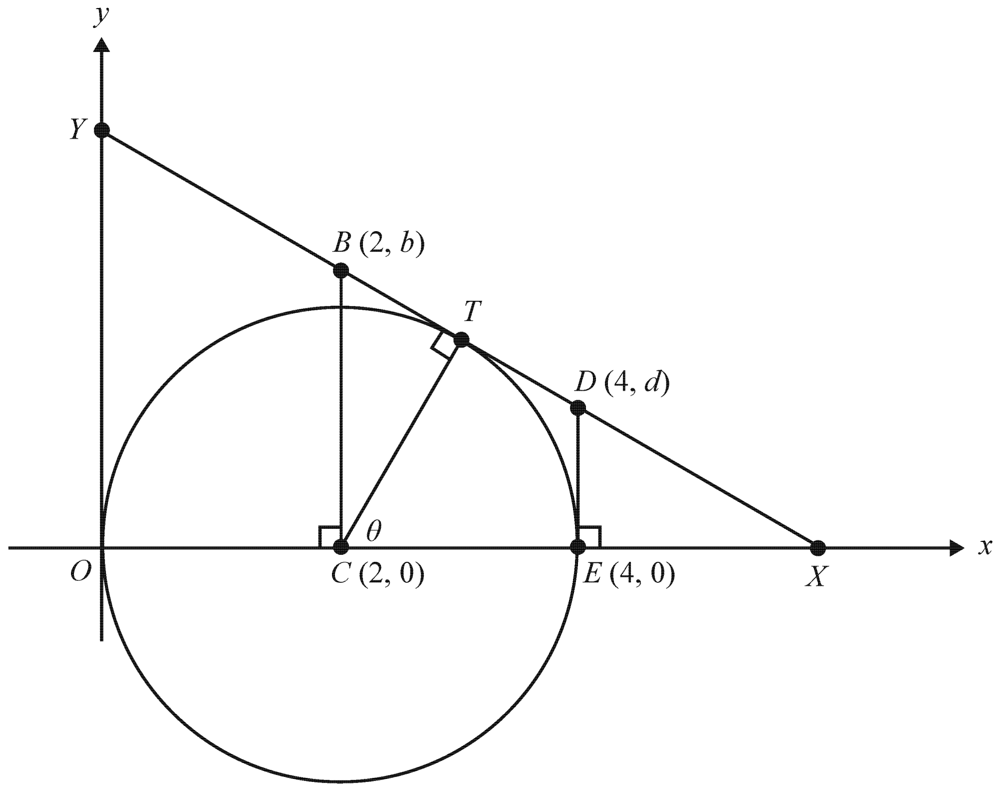
i. Find an expression for v in terms of u. 1 mark

ii. Find the **minimum** total shaded area and the value of u for which the area is a minimum. 2 marks

iii. Find the **maximum** total shaded area and the value of u for which the area is a maximum. 1 mark

QUESTION 7 – 2015 (HARD – HARD – AVERAGE – HARD – HARD)

The diagram below shows a point, T, on a circle. The circle has radius 2 and centre at the point C with coordinates (2, 0). The angle ECT is θ, where $0 < \theta \le \frac{\pi}{2}$.



The diagram also shows the tangent to the circle at T. This tangent is perpendicular to CT and intersects the x-axis at point X and the y-axis at point Y.

a. Find the coordinates of T in terms of θ. 1 mark

b. Find the gradient of the tangent to the circle at T in terms of θ. 1 mark

c. The equation of the tangent to the circle at T can be expressed as

$$\cos(\theta)x + \sin(\theta)y = 2 + 2\cos(\theta)$$

i. Point B, with coordinates $(2, b)$, is on the line segment XY.

Find b in terms of θ. 1 mark

ii. Point D, with coordinates $(4, d)$, is on the line segment XY.

Find d in terms of θ. 1 mark

d. Consider the trapezium *CEDB* with parallel sides of length b and d.

Find the value of θ for which the area of the trapezium *CEDB* is a minimum. Also find the minimum value of the area. 3 marks

SOLUTIONS

QUESTION 1

a.

$A = l \times w = 2ab$

when $x = a$, $y = b$ $\therefore b = 9 - 3a^2$

$\therefore A = 2a(9 - 3a^2)$

$= 18a - 6a^3$

Difficulty: 55% of students answered this question correctly
Average Score: 0.6 out of 1

b.

Let $\frac{dA}{da} = 0$

$18 - 18a^2 = 0$

$a^2 = 1$

$a = \pm 1$

$a = 1 \; (a > 0)$

max value of $A = 18a - 6a^3$

$= 18(1) - 6(1)^3$

$= 12$ square units

Difficulty: 26% of students answered this question correctly
Average Score: 1.2 out of 3

QUESTION 2

a. Volume = Area of triangle times y.

Use area of triangle $A=\frac{1}{2}bc\sin\left(A\right)$ for equilateral triangle

$V=\frac{1}{2}x^2\sin\left(60^0\right)y$ **(M1)**

$V=1000$

$1000=\frac{1}{2}x^2\frac{\sqrt{3}}{2}y$

$\therefore y=\frac{4000}{\sqrt{3}x^2}=\frac{4000\sqrt{3}}{3x^2}$ **(A1)**

Difficulty: 35% of students answered this question correctly
Average Score: 1.2 out of 2

b. Surface Area = 2 (area of triangle) + 3(rectangles)

$A=2\left(\frac{1}{2}x^2\frac{\sqrt{3}}{2}\right)+3\left(xy\right)$ where $y=\frac{4000\sqrt{3}}{3x^2}$ **(M1)**

$A=2\left(\frac{1}{4}\sqrt{3}x^2\right)+3\left(x\frac{4000\sqrt{3}}{3x^2}\right)$

Giving $A=\frac{\sqrt{3}x^2}{2}+\frac{4000\sqrt{3}}{x}$ **(M1)**

Difficulty: 31% of students answered this question correctly
Average Score: 0.9 out of 2

c. $A=\frac{\sqrt{3}x^2}{2}+\frac{4000\sqrt{3}}{x}$

$\frac{dA}{dx}=\sqrt{3}x-\frac{4000\sqrt{3}}{x^2}=0$ for minimum **(M1)**

$\sqrt{3}x=\frac{4000\sqrt{3}}{x^2}$

$\therefore \sqrt{3}x^3=4000\sqrt{3}$ **(M1)**

$\therefore x^3=4000$

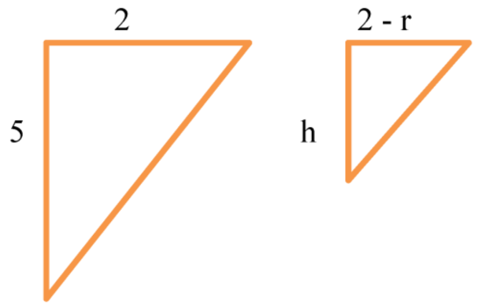
$\therefore x=\sqrt[3]{4000}$ **(A1)**

Alternative answers: $x=\sqrt[3]{4000}=4000^{\frac{1}{3}}=10\sqrt[3]{4}$

Difficulty: 30% of students answered this question correctly
Average Score: 1.3 out of 3

QUESTION 3

a. By similar triangles $\frac{2}{5} = \frac{2-r}{h} \Rightarrow 2h = 5(2-r)$ **(M1)**



$$h = \frac{10-5r}{2}$$ **(A1)**

Difficulty: 11% of students answered this question correctly
Average Score: 0.5 out of 2

b. $S = 2\pi rh + 2\pi r^2$ where $h = \frac{10-5r}{2}$

$$S = 2\pi r\left(\frac{10-5r}{2}\right) + 2\pi r^2$$

$$S = \pi r(10-5r) + 2\pi r^2$$

$$S = 10\pi r - 5\pi r^2 + 2\pi r^2$$

$$\therefore S = 10\pi r - 3\pi r^2$$

Difficulty: 47% of students answered this question correctly
Average Score: 0.5 out of 1

c. For maximum surface area:

$$S = 10\pi r - 3\pi r^2$$

$$\therefore \frac{dS}{dr} = 10\pi - 6\pi r = 0$$ **(M1)**

$$10\pi - 6\pi r = 0$$

$$\therefore r = \frac{5}{3}$$ **(A1)**

Difficulty: 10% of students answered this question correctly
Average Score: 0.5 out of 2

QUESTION 4

a. $\sin(\theta)=\dfrac{CD}{a}$ and $\cos(\theta)=\dfrac{BD}{a}$

$\therefore CD=a\sin(\theta)$ and $BD=a\cos(\theta)$ **(A2)**

Difficulty: 62% of students answered this question correctly
Average Score: 1.3 out of 2

b. $L=4+a+2a\cos(\theta)+a\sin(\theta)$ **(A1)**

Difficulty: 58% of students answered this question correctly
Average Score: 0.6 out of 1

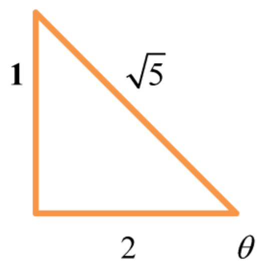
c. $\dfrac{dL}{d\theta}=a\cos(\theta)-2a\sin(\theta)$

$\dfrac{dL}{d\theta}=0 \Rightarrow a\cos(\theta)=2a\sin(\theta) \Rightarrow \tan(\theta)=\dfrac{1}{2}$ **(M1)**

But $\tan(\theta)=\dfrac{CD}{BD}=\dfrac{1}{2}$ so that $BD=2CD$ **(A1)**

Difficulty: 28% of students answered this question correctly
Average Score: 0.7 out of 2

d. $\tan(\theta)=\dfrac{1}{2} \Rightarrow \sin(\theta)=\dfrac{1}{\sqrt{5}}$ and $\cos(\theta)=\dfrac{2}{\sqrt{5}}$ since $0<\theta<\dfrac{\pi}{2}$



$L_{\max}=4+a+2a\times\dfrac{2}{\sqrt{5}}+a\times\dfrac{1}{\sqrt{5}}$ when $a=3\sqrt{5}$

$L_{\max}=4+3\sqrt{5}+12+3$ **(A1)**

$L_{\max}=19+3\sqrt{5}$

Difficulty: 5% of students answered this question correctly
Average Score: 0.1 out of 1

QUESTION 5

a.

$$\text{Area} = \frac{1}{2} \times b \times h = \frac{1}{2} \times x \times f(x)$$

$$= xe^{-\frac{x}{5}} \text{ squore units} \quad (A1)$$

Difficulty: 55% of students answered this question correctly
Average Score: 0.6 out of 1

b.

$$A = xe^{-x/5}$$

$$\frac{dA}{dx} = x\left(-\frac{1}{5}e^{-x/5}\right) + \left(e^{-x/5}\right) \quad (M1)$$

$$= e^{-x/5}\left(-\frac{x}{6} + 1\right)$$

$$= e^{-x/5}\left(1 - \frac{x}{5}\right)$$

$$\text{Let } \frac{dA}{dx} = 0 : \quad e^{-x/5}\left(1 - \frac{x}{5}\right) = 0$$

$$1 - \frac{x}{5} = 0$$

$$x = 5 \quad (A1)$$

$$A_{max} = xe^{-x/5} = 5e^{-1} \text{ square units } (A1)$$

Difficulty: 26% of students answered this question correctly
Average Score: 1.2 out of 3

c.

Upper limit: $y = 2e^{-x/5}$

$$\frac{1}{2} = 2e^{-x/5}$$

$$\frac{1}{4} = e^{-x/5}$$

$$\log_e\left(\frac{1}{4}\right) = \log_e e^{-x/5}$$

$$\log_e\left(\frac{1}{4}\right) = -\frac{x}{5}$$

$$5\log_e\left(\frac{1}{4}\right) = -x$$

$$\therefore x = 5\log_e 4 \quad (M1)$$

Area Shaded Region: $\frac{1}{2}\left(2+\frac{1}{2}\right)\times 5\log_e 4 - \int_0^{5\log_e 4} 2e^{-x/5}\,dx$ (M1)

$$= \frac{25}{2}\log_e 2 - \left[-10e^{-x/5}\right]^{5\log_e 4}$$

$$= \frac{25}{2}\log_e 2 - \frac{15}{2}$$

$$= \frac{5}{2}\left(5\log_e 2 - 3\right) \text{ units}^2 \quad (A1)$$

Difficulty: **7%** of students answered this question correctly
Average Score: 1 out of 3

QUESTION 6

a.

$$y = ax^2 + bx$$

$$\frac{dy}{dx} = 2ax + b \quad \text{(gradient of tangent)}$$

At $x = 2$, $\frac{dy}{dx} = 4a + b$

when $u = 6$: $m_{\text{line connecting Q and u}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{6-2} = -1$

$\therefore\ 4a + b = -1$ — ① (M1)

As $(2, 4)$ lies on $y = ax^2 + bx$:

$$4 = a(2)^2 + 2b$$

$$4 = 4a + 2b$$

$$2a + b = 2 \quad \text{— ②} \quad \text{(M1)}$$

$$4a + b = -1 \quad \text{— ①}$$

$$2a + b = 2 \quad \text{— ②}$$

① − ② : $2a = -3$

$$a = -\frac{3}{2} \quad \left(A\tfrac{1}{2}\right)$$

Sub $a = -\frac{3}{2}$ into ② :

$$2\left(-\frac{3}{2}\right) + b = 2$$

$$-3 + b = 2$$

$$\therefore b = 5 \quad \left(A\tfrac{1}{2}\right)$$

Difficulty: 30% of students answered this question correctly
Average Score: 1.5 out of 3

b. **(i)**

$$m_{VQ} = m_{UQ}$$

$$\frac{v-4}{0-2} = \frac{0-4}{u-2}$$

$$\frac{v-4}{-2} = \frac{-4}{u-2}$$

$$v-4 = \frac{8}{u-2}$$

$$v = 4 + \frac{8}{u-2} = \frac{4u}{u-2} \quad (A1)$$

Difficulty: 32% of students answered this question correctly
Average Score: 0.3 out of 1

(ii)

$$\text{Shaded Area} = \frac{1}{2}uv - 8$$

$$\text{As } v = \frac{4u}{u-2}: \quad \text{Area} = \frac{1}{2}u\left(\frac{4u}{u-2}\right) - 8$$

$$= \frac{2u^2}{u-2} - 8 \quad (m1)$$

$$\text{Let } \frac{dA}{du} = 0: \quad \frac{(u-2)(4u) - (2u^2)(1)}{(u-2)^2} = 0$$

$$4u^2 - 8u - 2u^2 = 0$$

$$2u^2 - 8u = 0$$

$$2u(u-4) = 0$$

$2u = 0$ OR $u - 4 = 0$

$u = 0$ $\quad$ $u = 4$

As $u > 0$ $\therefore u = 4$

$A = \frac{2u^2}{u-2} - 8 = \frac{2(4^2)}{4-2} - 8 = 8$ square units (local min)

Test endpoints as $\frac{5}{2} \leq u \leq 6$:

$u = \frac{5}{2}$, $A = \frac{2\left(\frac{5}{2}\right)^2}{\frac{5}{2} - 2} - 8 = 17$

$u = 6$, $A = \frac{2(6)^2}{6-2} - 8 = 10$

$\therefore A_{minimum} = 8$ square units (A1)

Difficulty: 9% of students answered this question correctly
Average Score: 0.4 out of 2

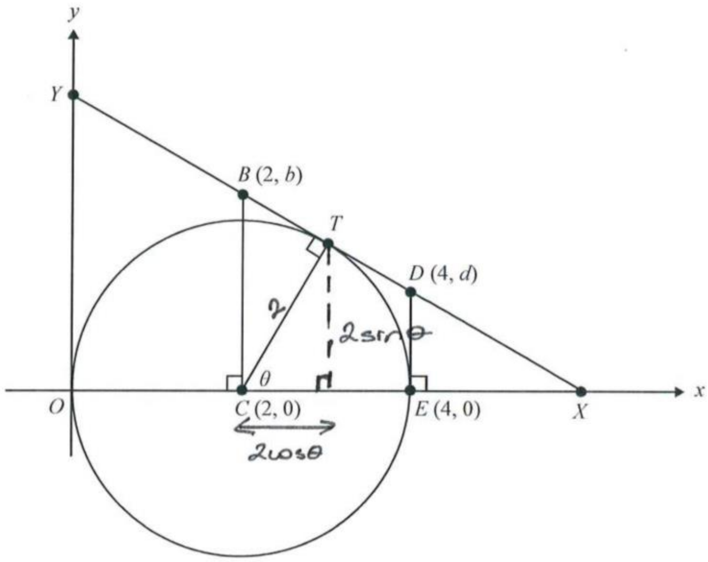
(iii)

maximum area = 17 square units (A1)

Difficulty: 8% of students answered this question correctly
Average Score: 0.1 out of 1

QUESTION 7

a.



$T = (2 + 2\cos\theta,\ 2\sin\theta)$

Difficulty: 20% of students answered this question correctly
Average Score: 0.2 out of 1

b.

$$m_{xy} = \frac{1}{m_{CT}}$$

$$m_{CT} = \frac{rise}{run} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$$

$$\therefore m_{tangent} = -\frac{1}{\tan\theta} \quad (A1)$$

Difficulty: 16% of students answered this question correctly
Average Score: 0.2 out of 1

c. **(i)**

$$x\cos\theta + y\sin\theta = 2 + 2\cos\theta$$

when $x = 2$, $y = b$:

$$2\cos\theta + b\sin\theta = 2 + 2\cos\theta$$

$$b\sin\theta = 2$$

$$b = \frac{2}{\sin\theta} \quad (A1)$$

Difficulty: 54% of students answered this question correctly
Average Score: 0.6 out of 1

(ii)

$$x\cos\theta + y\sin\theta = 2 + 2\cos\theta$$

when $x = 4$, $y = d$:

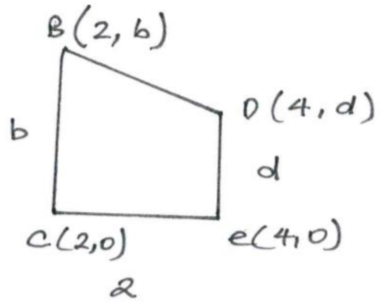
$$4\cos\theta + d\sin\theta = 2 + 2\cos\theta$$

$$d\sin\theta = 2 - 2\cos\theta$$

$$d = \frac{2 - 2\cos\theta}{\sin\theta} \quad (A1)$$

Difficulty: 47% of students answered this question correctly
Average Score: 0.5 out of 1

d.



$$A = \frac{1}{2}(b+d)2 = b+d$$

$$= \frac{2}{\sin\theta} + \frac{2-2\cos\theta}{\sin\theta}$$

$$= \frac{4-2\cos\theta}{\sin\theta}$$

$$\frac{dA}{d\theta} = \frac{\sin\theta(2\sin\theta)-(4-2\cos\theta)\cos\theta}{(\sin\theta)^2} \quad (M1)$$

$$= \frac{2\sin^2\theta - 4\cos\theta + 2\cos^2\theta}{\sin^2\theta}$$

$$= \frac{2(\sin^2\theta + \cos^2\theta) - 4\cos\theta}{\sin^2\theta}$$

$$= \frac{2-4\cos\theta}{\sin^2\theta}$$

Let $\frac{dA}{d\theta} = 0$: $\quad 2 - 4\cos\theta = 0$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3} \quad (A1)$$

$$\text{Area minimum} = \frac{4-2\cos\theta}{\sin\theta} = \frac{4-2\cos\frac{\pi}{3}}{\sin\pi/3}$$

$$= \frac{4-1}{\sqrt{3}/2}$$

$$= 2\sqrt{3} \text{ units squared } (A1)$$

Difficulty: 11% of students answered this question correctly
Average Score: 0.6 out of 3