Chapter 12 Vector Functions

Exercise 12A Vector functions

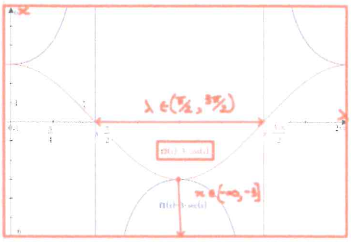
→ Deriving cartesian equation of path : $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$

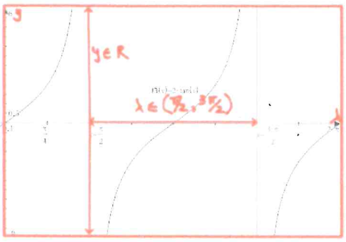
- $x = x(t)$, $y = y(t)$ → parametric eqns (combined by eliminating parameter 't')
- Domain : range of function $x = x(t)$
- Range : range of function $y = y(t)$
- e.g. $\underset{\sim}{v}(t) = (v\cos(\theta))\underset{\sim}{i} + (v\sin(\theta) - gt)\underset{\sim}{j}$

eg Find the cartesian equation which corresponds to the following :—
$\underset{\sim}{r}(\lambda) = 3\sec(\lambda)\underset{\sim}{i} + 2\tan(\lambda)\underset{\sim}{j}$, $\lambda \in (\frac{\pi}{2}, \frac{3\pi}{2})$.
State the domain and range and sketch the graph.

So $x = 3\sec(\lambda)$, $\lambda \in (\frac{\pi}{2}, \frac{3\pi}{2}) \to x \in (-\infty, 3]$
$y = 2\tan(\lambda)$, $\lambda \in (\frac{\pi}{2}, \frac{3\pi}{2}) \to y \in R$

Now $\sec(\lambda) = \frac{x}{3}$
$\tan(\lambda) = \frac{y}{2}$





Now $\sec(\lambda) = \frac{x}{3}$ recap...
$\tan(\lambda) = \frac{y}{2}$

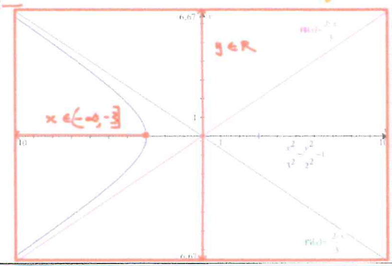
Now $1 + \tan^2(\lambda) = \sec^2(\lambda)$ — (*)
Sub for $\sec(\lambda)$, $\tan(\lambda)$ from above into (*) :—
→ $1 + (\frac{y}{2})^2 = (\frac{x}{3})^2$
→ $\boxed{\frac{x^2}{9} - \frac{y^2}{4} = 1}$

Domain is ~~(−∞, 3]~~ $(-\infty, -3]$
Range is R

Now for the sketch graph :—

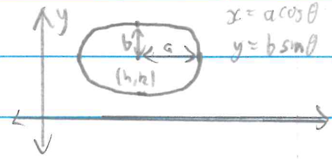
This is a ")(" shaped hyperbola.
x-ints :— $y = 0$, → $x = \pm 3$
Centre is at the origin.
Asymptotes are $y = \pm\frac{2x}{3}$

→ LH Branch ONLY allowed !



Ellipses : $\left(\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\right)$

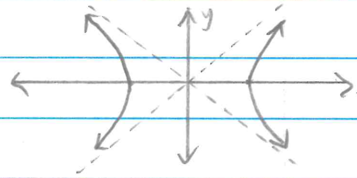
$x = a\cos\theta$
$y = b\sin\theta$



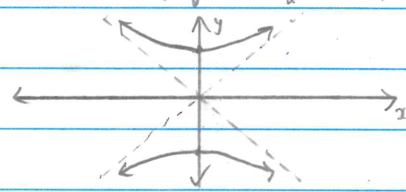
Hyperbolas :

Centre : (h,k), Asymptotes : $y = \pm\frac{b}{a}(x-h) + k$

- East West $\left(\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1\right)$



- North South $\left(\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1\right)$



Exercise 12B Position vectors as a function of time

→ Position vector : $\overrightarrow{OP} = \underset{\sim}{r} = x\underset{\sim}{i} + y\underset{\sim}{j}$

- $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j}$ (2D)
- $\underset{\sim}{r}(t) = x(t)\underset{\sim}{i} + y(t)\underset{\sim}{j} + z(t)\underset{\sim}{k}$ (3D)
- changes with time
- actual trajectory of particle in real time
- convert parametric eqn into cartesian eqn – 'time' info lost

Example 5

An object moves along a path where the position vector is given by $r(t) = \cos ti + \sin tj + 2k, t \geq 0$

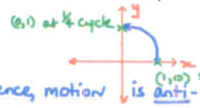
Describe the motion of the object

$\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + 2\underset{\sim}{k}, t \geq 0$

- $\underset{\sim}{r}(t)$ is described in 3 dimensions, with components in the $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ directions.

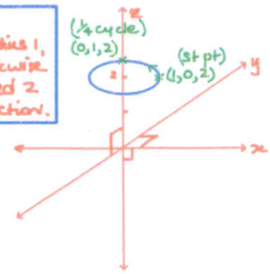
The $\underset{\sim}{k}$ component does not change with time.
So the particle is permanently displaced by 2 units in the +ve $\underset{\sim}{k}$ direction, at right angles to the x-y plane.

Regarding the x-y plane:—
$x = \cos(t)$ —①
$y = \sin(t)$ —②
Now $\sin^2(t) + \cos^2(t) = 1$ —(*)
Sub for x and y from ①+② into (*):—
→ $x^2 + y^2 = 1$
→ motion is circular, centre (0,0), and radius 1.
Now for the direction of motion:—
$x = \cos(t)$, $y = \sin(t)$
When $t = 0$, $x = 1$, $y = 0$ → starts at (1,0)
After a ¼ cycle, $t = \pi/2$.
When $t = \pi/2$, $x = \cos(\pi/2) = 0$
$y = \sin(\pi/2) = 1$
→ After a ¼ cycle, position is (0,1).



Hence, motion is anti-clockwise.

Circular motion in the x-y plane, radius 1, centre (0,0), anti-clockwise from (1,0), displaced 2 units in the $+\underset{\sim}{k}$ direction.



eg. The vector equation $r(t) = -\cos(2\pi t)i + \sin(2\pi t)j$ again describes a particle moving around the circle but with the following features:

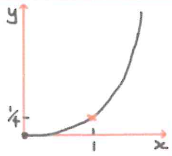
- at time $t = 0$, the particle is at $(-1, 0)$
- the particle moves on the curve with equation $x^2 + y^2 = 1$
- the particle moves in a clockwise direction
- the particle moves around the circle with a period of one unit

Example 4

Sketch the path of a particle where the position at time t is given by $r(t) = 2ti + t^2j, t \geq 0$.

$\underset{\sim}{r}(t) = 2t\underset{\sim}{i} + t^2\underset{\sim}{j}, t \geq 0$

So $x = 2t$ —①
$y = t^2$ —②
Now from ①, $t = x/2$
Sub $t = x/2$ into eq^n ②:—
② → $y = (x/2)^2$
→ $\boxed{y = \frac{x^2}{4}, x \geq 0}$ (if $t \geq 0$)



NB! Say for example $x = \frac{t}{2}$; $y = \frac{t^2}{16}$
This has the same cartesian eq^n, but the travel along that path occurs at a different rate to the first example opposite.
In fact, there are infinitely many possible parametric equations for x and y which give the same cartesian result.

→ Cartesian to Parametric (linear rate assumption)

- e.g. $(x-1)^2 + \frac{y^2}{4} = 4$

$t = 0 \to (1,4)$, $t = \frac{\pi}{2} \to (3,0)$

$m = \frac{3-1}{\frac{\pi}{2} - 0} = \frac{4}{\pi}$

t	x
0	1
$\frac{\pi}{2}$	0

$x = \frac{4}{\pi}t + 1$

$y = \pm 2\sqrt{x(2-x)}$

$= \pm 2\sqrt{2\left(\frac{4}{\pi}t + 1\right) - \left(\frac{4}{\pi}t + 1\right)^2}$

$\therefore \underset{\sim}{r}(t) = \left(\frac{4}{\pi}t + 1\right)\underset{\sim}{i} + 2\sqrt{2\left(\frac{4}{\pi}t + 1\right) - \left(\frac{4}{\pi}t + 1\right)^2}\ \underset{\sim}{j}$

→ Conditions for two particles to meet

- $\underset{\sim}{r}(t) = x_r(t)\underset{\sim}{i} + y_r(t)\underset{\sim}{j}$ and $\underset{\sim}{s}(t) = x_s(t)\underset{\sim}{i} + y_s(t)\underset{\sim}{j}$
- Collide at same place at same time if:

$\underset{\sim}{r}(t) = \underset{\sim}{s}(t)$, $x_r(t) = x_s(t)$, $y_r(t) = y_s(t)$, $t_r = t_s$

- Cross paths at same place but different times if:

$\underset{\sim}{r}(t) \neq \underset{\sim}{s}(t)$, $x_r(t) = x_s(t)$, $y_r(t) = y_s(t)$, $t_r \neq t_s$

- Complex example

Example 6

The motion of two particles is given by the vector equations $r_1(t) = (2t - 3)i + (t^2 + 10)j$ and $r_2(t) = (t + 2)i + 7tj$, where $t \geq 0$. Find:

a the point at which the particles collide
b the points at which the two paths cross
c the distance between the particles when $t = 1$

(a) $\underset{\sim}{r}_1(t) = (2t-3)\underset{\sim}{i} + (t^2+10)\underset{\sim}{j}$
$\underset{\sim}{r}_2(t) = (t+2)\underset{\sim}{i} + 7t\underset{\sim}{j}$
If particles collide, they are both at the same place at the same time.
So $x_1 = x_2$, $y_1 = y_2$ and $t_1 = t_2$
→ $2t - 3 = t + 2$ —①
and $t^2 + 10 = 7t$ —②
From ①, $t = 5$ ($x_1 = x_2$ at $t = 5$)
From ②, $t^2 - 7t + 10 = 0$
i.e. $(t-5)(t-2) = 0$ → $t = 5$ or 2
→ ($y_1 = y_2$ at $t = 2, 5$)
So for $x_1 = x_2$, $y_1 = y_2$ to both be satisfied at the same time, $t = 5$.
Checking answer:—
$\underset{\sim}{r}_1(5) = 7\underset{\sim}{i} + 35\underset{\sim}{j}$; $\underset{\sim}{r}_2(5) = 7\underset{\sim}{i} + 35\underset{\sim}{j}$ ✓✓
$\boxed{\text{So point of collision is } 7\underset{\sim}{i} + 35\underset{\sim}{j}}$

(b) If particles cross paths, they are at the same place, but at different times.
So $x_1 = x_2$, $y_1 = y_2$ and $t_1 \neq t_2$
→ $2t_1 - 3 = t_2 + 2$ —①
and $t_1^2 + 10 = 7t_2$ —②
From ①, $t_2 = 2t_1 - 5$ Sub for t_2 into ②:—
② → $t_1^2 + 10 = 7(2t_1 - 5)$
→ $t_1^2 + 10 = 14t_1 - 35$
→ $t_1^2 - 14t_1 + 45 = 0$
→ $(t_1 - 5)(t_1 - 9) = 0$
→ $t_1 = 5$ or 9 → $t_2 = 5$ or 13 (respectively)

So when $x_1 = x_2$ and $y_1 = y_2$,
$t_1 = 5$ and $t_2 = 5$ → collision, found in (a)
$t_1 = 9$ and $t_2 = 13$ → paths cross at different times
Check answer:—
$\underset{\sim}{r}_1(9) = (2\times9-3)\underset{\sim}{i} + (9^2+10)\underset{\sim}{j}$
$= 15\underset{\sim}{i} + 91\underset{\sim}{j}$
$\underset{\sim}{r}_2(13) = (13+2)\underset{\sim}{i} + 7\times13\underset{\sim}{j}$
$= 15\underset{\sim}{i} + 91\underset{\sim}{j}$ ✓✓

$\boxed{\text{So the point of paths crossing is } 15\underset{\sim}{i} + 91\underset{\sim}{j}}$

(c) When $t = 1$,
$\underset{\sim}{r}_1(1) = (2\times1-3)\underset{\sim}{i} + (1^2+10)\underset{\sim}{j} = -\underset{\sim}{i} + 11\underset{\sim}{j}$
$\underset{\sim}{r}_2(1) = (1+2)\underset{\sim}{i} + 7\times1\underset{\sim}{j} = 3\underset{\sim}{i} + 7\underset{\sim}{j}$

By Pythagoras' theorem, distance between $\underset{\sim}{r}_1(1)$ and $\underset{\sim}{r}_2(1)$ $= \sqrt{(-1-3)^2 + (11-7)^2}$
$= \sqrt{(-4)^2 + (4)^2}$
$= \boxed{4\sqrt{2}}$

Exercise 12C Vector calculus

→ General form: $r(t) = x(t)\, i + y(t)\, j$

① $\dot{r}(t) = \dot{x}(t)\, i + \dot{y}(t)\, j = \frac{dx}{dt} i + \frac{dy}{dt} j$

② $\ddot{r}(t) = \ddot{x}(t)\, i + \ddot{y}(t)\, j = \frac{d^2x}{dt^2} i + \frac{d^2y}{dt^2} j$

③ $r(t) = \int \dot{r}(t)\, dt = \int \dot{x}(t)\, dt\; i + \int \dot{y}(t)\, dt\; j + C$

④ $\dot{r}(t) = \int \ddot{r}(t)\, dt = \int \ddot{x}(t)\, dt\; i + \int \ddot{y}(t)\, dt\; j + C$

Exercise 12D Velocity and acceleration for motion along a curve

→ Velocity: $v(t) = \dot{r}(t) = \dot{x}(t)\, i + \dot{y}(t)\, j$ (N.B. velocity = direction of motion)

→ Acceleration: $a(t) = \dot{v}(t) = \ddot{r}(t) = \ddot{x}(t)\, i + \ddot{y}(t)\, j$ ∴ Up: $a = -g - kv^2$

Down: $a = g - kv^2$

→ Speed: $|\dot{r}(t)|$

→ Distance between two points on curve: $|r(t_1) - r(t_0)|$

→ Distance travelled along a curve:

· $L = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

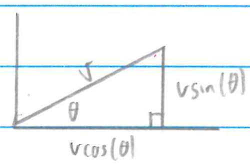
→ General equation for 2D vectors + kinematics

· $a(t) = -g\, j$

· $v(t) = v\cos(\theta)\, i + (v\sin(\theta) - gt)\, j$

· $r(t) = (v\cos(\theta)\, t)\, i\, (v\sin(\theta)\, t - \frac{1}{2} g t^2)\, j + c$

(where $r(0) = c$)



* N.B. calculating general expressions in time function: $n \in Z^+ \cup \{0\}$

(not $n \in Z$)

VCAA 2007 Exam 1 Question 9 – Average Marks 0.9/3

A particle moves in the cartesian plane with position vector $r = xi + yj$ where x and y are functions of t. If its velocity vector is $v = -yi + xj$, find the acceleration vector of the particle in terms of the position vector r.

①

$$v = \frac{dr}{dt} = \frac{dx}{dt} i + \frac{dy}{dt} j = -y\,i + x\,j$$

$$\Rightarrow \frac{dx}{dt} = -y \qquad \frac{dy}{dt} = x$$

②

$$\frac{dv}{dt} = -\frac{dy}{dt} i + \frac{dx}{dt} j$$

$$= -x\,i - y\,j$$

$$= -(x\,i + y\,j) \; \boxed{= -r}$$

An ant (A) and a beetle (B) move in a Cartesian plane so that at any time $t \geq 0$ their position vectors are

$\underset{\sim}{r}_A = 4t\underset{\sim}{i} + t\underset{\sim}{j}$

$\underset{\sim}{r}_B = (8 - 8\sin(\alpha t))\underset{\sim}{i} + 8\cos(\alpha t)\underset{\sim}{j}$, where α is a positive constant, $\underset{\sim}{i}$ is a unit vector in the positive x direction and $\underset{\sim}{j}$ is a unit vector in the positive y direction.

a. Find the speeds of both insects, A and B. (3 marks)

$\dot{\underset{\sim}{r}}_A(t) = 4\underset{\sim}{i} + \underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}_A(t)| = \sqrt{4^2+1^2} = \boxed{\sqrt{17}}$

$\dot{\underset{\sim}{r}}_B(t) = -8\alpha\cos(\alpha t)\,\underset{\sim}{i} - 8\alpha\sin(\alpha t)\,\underset{\sim}{j}$

$|\dot{\underset{\sim}{r}}_B(t)| = \sqrt{(-8\alpha\cos(\alpha t))^2 + (-8\alpha\sin(\alpha t))^2}$

$= \sqrt{64\alpha^2} = \boxed{8\alpha}$ (since $\alpha > 0$)

c. Find the coordinates of the point(s) where the paths of the insects intersect. (2 marks)

Sub $y = \frac{x}{4}$ into $(x-8)^2 + y^2 = 64$

$(x-8)^2 + \left(\frac{x}{4}\right)^2 = 64$

$x^2 - 16x + 64 + \frac{x^2}{16} = 64 \rightarrow 17x^2 - 256x = 0$

$x(17x - 256) = 0$

$\rightarrow x = 0$ or $x = \frac{256}{17}$

$\therefore \boxed{(0,0),\ \left(\frac{256}{17}, \frac{64}{17}\right)}$

d. Find the smallest positive value of α for which the insects collide. Give your answer correct to two decimal places. (4 marks)

Ant (A) is at $\left(\frac{256}{17}, \frac{64}{17}\right)$ when $t = \frac{64}{17}$

(B): $\frac{256}{17} = 8 - 8\sin(\alpha t)$ $\quad \frac{64}{17} = 8\cos(\alpha t)$

When $t = \frac{64}{17}$, $\frac{256}{17} = 8 - 8\sin\left(\frac{64}{17}\alpha\right)$ ... ①

$\frac{64}{17} = 8\cos\left(\frac{64}{17}\alpha\right)$ ... ②

Solve ①: $\alpha = 1.12, 1.38, 2.79, 3.03$

Solve ②: $\alpha = 0.29, 1.38, 1.96, 3.05$

$\boxed{\alpha = 1.38}$ (2 d.p.s)

b. Find the Cartesian equations of the paths of both insects and sketch the paths on the same set of axes, clearly indicating their starting positions and direction of motion. (5 marks)

(A) $x = 4t$, $y = t$

$\rightarrow \boxed{y = \frac{x}{4},\ x \geq 0}$

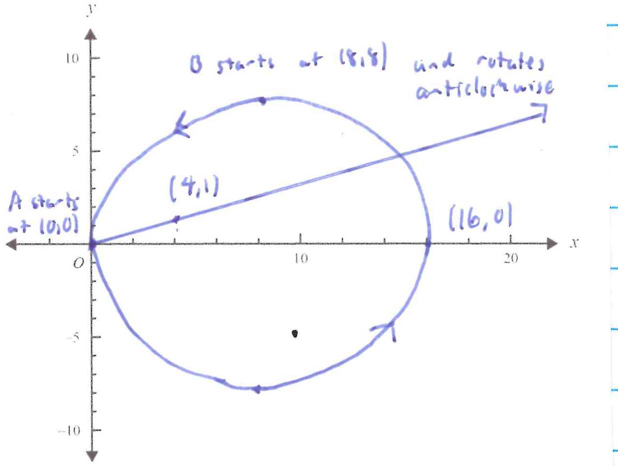
(B) $x = 8 - 8\sin(\alpha t) \rightarrow \sin(\alpha t) = \frac{8-x}{8}$

$y = 8\cos(\alpha t) \rightarrow \cos(\alpha t) = \frac{y}{8}$

$\sin^2(\alpha t) + \cos^2(\alpha t) = 1$

$\therefore \frac{(8-x)^2}{64} + \frac{y^2}{64} = 1$

$\boxed{(x-8)^2 + y^2 = 64,\ 0 \leq x \leq 16}$



e. Given that the insects collide, find the distance travelled by each insect until the collision occurs. Give your answer correct to one decimal place. (2 marks)

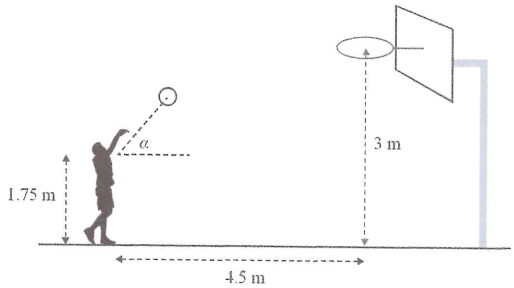
(A) distance $= \sqrt{17} \times \frac{64}{17} = 15.5$ (1dp)

(B) distance $= 8\alpha \times \frac{64}{17} = 41.6$ (1dp)

(when $\alpha = 1.38$)

VCAA NH 2018 Exam 2 Q4

A basketball player aims to throw a basketball through a ring, the centre of which is at a horizontal distance of 4.5 m from the point of release of the ball and 3 m above floor level. The ball is released at a height of 1.75 m above floor level, at an angle of projection α to the horizontal and at a speed of V ms^{-1}. Air resistance is assumed to be negligible.



The position vector of the centre of the ball at any time, t seconds, for $t \geq 0$, relative to the point of release is given by $\underset{\sim}{r}(t) = Vt\cos(\alpha)\underset{\sim}{i} + \left(Vt\sin(\alpha) - 4.9t^2\right)\underset{\sim}{j}$, where $\underset{\sim}{i}$ is a unit vector in the horizontal direction of motion of the ball and $\underset{\sim}{j}$ is a unit vector vertically up. Displacement components are measured in metres.

a. For the player's first shot at goal, $V = 7$ ms^{-1} and $\alpha = 45°$.

i. Find the time, in seconds, taken for the ball to reach its maximum height. Give your answer in the form $\frac{a\sqrt{b}}{c}$, where a, b and c are positive integers. 2 marks

$\underset{\sim}{r}(t) = 7t\cos(45°)\underset{\sim}{i} + (7t\sin(45°) - 4.9t^2)\underset{\sim}{j}$

$= \frac{7\sqrt{2}}{2}t\,\underset{\sim}{i} + \left(\frac{7\sqrt{2}}{2}t - \frac{49}{10}t^2\right)\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \frac{7\sqrt{2}}{2}\underset{\sim}{i} + \left(\frac{7\sqrt{2}}{2} - \frac{49}{5}t\right)\underset{\sim}{j}$

Max height when $\frac{7\sqrt{2}}{2} - \frac{49}{5}t = 0$

$\therefore \boxed{t = \frac{5\sqrt{2}}{14}\text{ s}}$

ii. Find the maximum height, in metres, above floor level, reached by the centre of the ball. 2 marks

$y(t) = \frac{7\sqrt{2}}{2}t - \frac{49}{10}t^2$ (height)

Maximum height when $t = \frac{5\sqrt{2}}{14}$

$y(t) = \frac{7\sqrt{2}}{2}\left(\frac{5\sqrt{2}}{14}\right) - \frac{49}{10}\left(\frac{5\sqrt{2}}{14}\right)^2 = \frac{5}{4}$

, Max height above floor level

$= \frac{5}{4} + 1.75 = \boxed{3\text{ m}}$

iii. Find the distance of the centre of the ball from the centre of the ring one second after release. Give your answer in metres, correct to two decimal places. 2 marks

$\underset{\sim}{r}(1) = \frac{7\sqrt{2}}{2}\underset{\sim}{i} + \left(\frac{7\sqrt{2}}{2} - \frac{49}{10}\right)\underset{\sim}{j}$

Vertical distance from relative point of launch to ring $= 3 - 1.75 = 1.25$ m

Distance $= \sqrt{\left(\frac{7\sqrt{2}}{2} - 4.5\right)^2 + \left(\frac{7\sqrt{2}}{2} - \frac{49}{10} - 1.25\right)^2}$

$= \boxed{1.28\text{ m}}$ (2dps)

b. For the player's second shot at goal, $V = 10$ ms^{-1}.

Find the possible angles of projection, α, for the centre of the ball to pass through the centre of the ring. Give your answers in degrees, correct to one decimal place. 3 marks

$\underset{\sim}{r}(t) = 10t\cos(\alpha)\underset{\sim}{i} + (10t\sin(\alpha) - \frac{49}{10}t^2)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = 4.5\underset{\sim}{i} + 1.25\underset{\sim}{j}$ $(x(t) = 4.5,\ y(t) = 1.25)$

$\rightarrow 10t\cos(\alpha) = 4.5$ --- ①

$10t\sin(\alpha) - \frac{49}{10}t^2 = 1.25$ --- ②

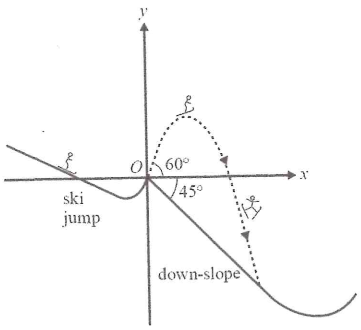
Solve ① and ② simultaneously for $0° < \alpha < 90°$ or for $0° < \alpha < 45°$ and $45° < \alpha < 90°$ on CAS,

$\rightarrow \boxed{\alpha = 29.7°,\ \alpha = 75.8°}$

VCAA 2016 (Sample) Exam 2 Q4

A skier accelerates down a slope and then skis up a short ski jump, as shown below. The skier leaves the jump at a speed of 12 ms^{-1} and at an angle of 60° to the horizontal. The skier performs various gymnastic twists and lands on a straight-line section of the 45° down-slope T seconds after leaving the jump.

Let the origin O of a cartesian coordinate system be at the point where the skier leaves the jump, with $\underset{\sim}{i}$ a unit vector in the positive x direction and $\underset{\sim}{j}$ a unit vector in the positive y direction. Displacements are measured in metres and time in seconds.



a. Show that the initial velocity of the skier when leaving the jump is $6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$. 1 mark

$\dot{\underset{\sim}{r}}(0) = 12\cos(60°)\underset{\sim}{i} + 12\sin(60°)\underset{\sim}{j}$

$= 12\left(\frac{1}{2}\right)\underset{\sim}{i} + 12\left(\frac{\sqrt{3}}{2}\right)\underset{\sim}{j}$

$= \boxed{6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}}$ (Shown)

b. The acceleration of the skier, t seconds after leaving the ski jump, is given by

$$\ddot{\underset{\sim}{r}}(t) = -0.1t\underset{\sim}{i} - (g - 0.1t)\underset{\sim}{j},\ 0 \le t \le T$$

Show that the position vector of the skier, t seconds after leaving the jump, is given by

$$\underset{\sim}{r}(t) = \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6t\sqrt{3} - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\underset{\sim}{j},\ 0 \le t \le T$$

3 marks

$\dot{\underset{\sim}{r}}(t) = \int \ddot{\underset{\sim}{r}}(t)\,dt = -\frac{1}{20}t^2\underset{\sim}{i} - \left(gt - \frac{1}{20}t^2\right)\underset{\sim}{j} + \underset{\sim}{c}$

Sub $\dot{\underset{\sim}{r}}(0) = 6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$, $\underset{\sim}{c} = 6\underset{\sim}{i} + 6\sqrt{3}\underset{\sim}{j}$

$\dot{\underset{\sim}{r}}(t) = \left(6 - \frac{1}{20}t^2\right)\underset{\sim}{i} + \left(6\sqrt{3} - gt + \frac{1}{20}t^2\right)\underset{\sim}{j}$

$\underset{\sim}{r}(t) = \int \dot{\underset{\sim}{r}}(t)\,dt$

$= \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6\sqrt{3}t - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right) + \underset{\sim}{d}$

Sub $\underset{\sim}{r}(0) = 0\underset{\sim}{i} + 0\underset{\sim}{j}$, $\underset{\sim}{d} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

$\therefore \underset{\sim}{r}(t) = \left(6t - \frac{1}{60}t^3\right)\underset{\sim}{i} + \left(6\sqrt{3}t - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)\underset{\sim}{j}$

(Shown) $(0 \le t \le T)$

c. Show that $T = \frac{12}{g}\left(\sqrt{3}+1\right)$. 3 marks

Gradient $= \tan(-45°) = -1$

$\therefore 6t - \frac{1}{60}t^3 = -\left(6\sqrt{3}t - \frac{1}{2}gt^2 + \frac{1}{60}t^3\right)$

$\frac{1}{2}gt^2 - (6\sqrt{3} + 6)t = 0$

$\frac{1}{2}t\left(gt - 12(\sqrt{3}+1)\right) = 0$

$t = 0$ (reject) or $gt - 12(\sqrt{3}+1) = 0$ (accept)

$gt = 12(\sqrt{3}+1) \rightarrow t = \frac{12}{g}(\sqrt{3}+1)$

Let $t = T$, $\rightarrow \boxed{T = \frac{12}{g}(\sqrt{3}+1)}$ (Shown)