(ii) P, A and B are collinear

$\to \overrightarrow{OP} = m\overrightarrow{OA} + (1-m)\overrightarrow{OB}, \quad m \in R\setminus\{0\}$

$= m(3\underset{\sim}{i} + 4\underset{\sim}{k}) + (1-m)(\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k})$

$= (2m+1)\underset{\sim}{i} + (2-2m)\underset{\sim}{j} + (6m-2)\underset{\sim}{k}$

$OP \perp AB \to \overrightarrow{OP} \cdot \overrightarrow{AB} = 0$

$-2(2m+1) + 2(2-2m) - 6(6m-2) = 0$

$-44m + 14 = 0 \to m = \frac{7}{22}$

$\therefore \overrightarrow{OP} = \boxed{\frac{18}{11}\underset{\sim}{i} + \frac{15}{11}\underset{\sim}{j} - \frac{1}{11}\underset{\sim}{k}}$

(iii) $\overrightarrow{OP} = (2m+1)\underset{\sim}{i} + (2-2m)\underset{\sim}{j} + (6m-2)\underset{\sim}{k}$

$$\frac{\overrightarrow{OA} \cdot \overrightarrow{OP}}{|\overrightarrow{OA}|} = \frac{\overrightarrow{OB} \cdot \overrightarrow{OP}}{|\overrightarrow{OB}|}$$

$$\frac{3(1+2m) + 4(6m-2)}{5} = \frac{1+2m+2(2-2m)-2(6m-2)}{3}$$

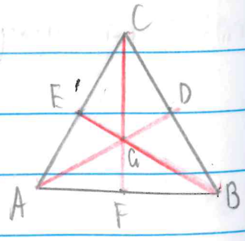
$\to m = \frac{3}{8}$

$\therefore \overrightarrow{OP} = \boxed{\frac{7}{4}\underset{\sim}{i} + \frac{5}{4}\underset{\sim}{j} + \frac{1}{4}\underset{\sim}{k}}$

Exercise 2F Geometric Proof

→ Useful properties – characterise geometric shape used in vector proof (underlined all blue)

- For $k \in R^+$ → vector $k\underset{\sim}{a}$ same direction as $\underset{\sim}{a}$ with magnitude $k|\underset{\sim}{a}|$
  → vector $-k\underset{\sim}{a}$ opposite direction as $\underset{\sim}{a}$ with magnitude $k|\underset{\sim}{a}|$
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ parallel → $\underset{\sim}{b} = k\underset{\sim}{a}$ for some $k \in R\setminus\{0\}$ ($\underset{\sim}{a}$ and $\underset{\sim}{b}$ are non-zero vectors)
- $\underset{\sim}{a}$ and $\underset{\sim}{b}$ parallel with at least one point in common
  - $\underset{\sim}{a}$ and $\underset{\sim}{b}$ line on same straight line (collinear) – $\overrightarrow{AB} \parallel \overrightarrow{BC}$ (B common point)
  - e.g. $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in R\setminus\{0\}$ → A + B + C collinear
- two non-zero vectors perpendicular → $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$
- Concentric lines → all pass through given point
  - If medians of $\triangle ABC$, vectors $\overrightarrow{AD}$, $\overrightarrow{BE}$ and $\overrightarrow{CF}$ are concentric at G, then after finding intersection point of $\overrightarrow{AD} + \overrightarrow{BE}$, $\overrightarrow{CG} + \overrightarrow{GF}$ would be parallel



- Side lengths squared $|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}$
- Lines of equal length. Lengths of vectors are equal → e.g. $|\underset{\sim}{a}| = |\underset{\sim}{b}|$

→ Steps for simple geometric theorems

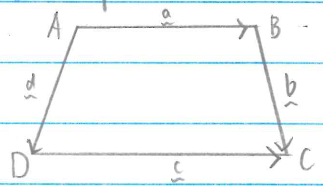
① Draw clear + well labeled diagram

② Clear state in vector form all info (explicit + implicit) given in problem

· Linear dependent → Perpendicular / Parallel → Eliminate all other variables

→ Simple shapes with vector constraints

① Trapezium ($\overrightarrow{AB} \parallel \overrightarrow{DC}$)



A: $b = d$ X $b \neq d$

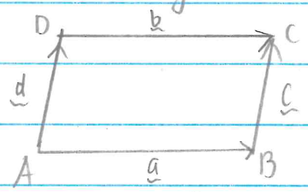
B: $a = c$ X $a = kc$, not $a = c$

C: $a \neq b$ X True, but not necessary

D: $|a| \neq |c|$ X Must also specify that $a \parallel c$

→ E: $c = ka$ ✓ True → one set of parallel sides

② Parallelogram



Both sets of opposite sides parallel

A: $|a| = |b|$ X Must be parallel

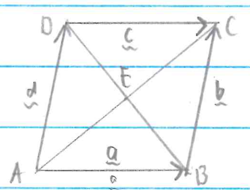
→ B: $c = d$ ✓ Means parallel and equal length

C: $a = b$ and $c = d$ X More than needed

D: $(a + c) \cdot (a - d) = 0$ X Diagonals not perpendicular

E: $|a| = |b|$ and $c = d$ X More than needed

③ Rhombus



Both sets of opposite sides parallel

Adjacent sides equal in length

A: $a = c$ X enough for a parallelogram only

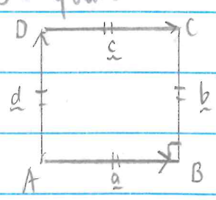
B: $a = c$ and $b = d$ X (over) specifies parallelogram, but insuff. for rhombus

→ C: $a = c$ and $|a| = |b|$ ✓

D: $(a + b) \cdot (d - a) \neq 0$ X Dot product for rhombus (and square) $= 0$

E: $\overrightarrow{AE} = \overrightarrow{EC}$ and $\overrightarrow{BE} = \overrightarrow{FD}$ X Both diagonals are bisected by each other in all parallelograms

④ Square



A: $a \cdot b = 0$ X Only means $a \perp b$. Other corners could be non-right angles

B: $a \cdot b = 0$ and $b \cdot c$ X Means rt. angles at B and C only

C: $a = c$ and $|a| = |b|$ X $a = c$ → parallelogram
$|a| = |b|$ → rhombus

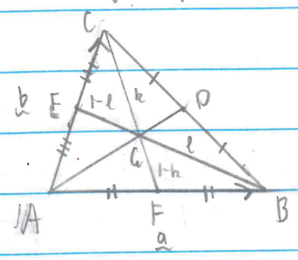
D: $|a| = |b| = |c|$ X guarantees rhombus, not a square

→ E: $b = d$ and $|b| = |c|$ and $c \cdot d = 0$ ✓
(parallelogram) (rhombus) (square)

→ Complex example

Prove that the medians of a triangle are concentric and that they are divided into lengths in ratio 1:2 by point of intersection.

→ Let $\overrightarrow{CG} = h\overrightarrow{CF}$ and $\overrightarrow{BG} = \ell\overrightarrow{BE}$.



Let $\overrightarrow{AB} = \underset{\sim}{a}$ and $\overrightarrow{AB} = \underset{\sim}{b}$

Let G be point of intersection of median's CF and BE.

$\overrightarrow{CF} = \overrightarrow{CA} + \overrightarrow{AF} = -\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}$

$\overrightarrow{CG} = \overrightarrow{CE} + \overrightarrow{EG} = -\frac{1}{2}\underset{\sim}{b} + (1-\ell)\overrightarrow{EB}$

$= -\frac{1}{2}\underset{\sim}{b} + (1-\ell)(-\frac{1}{2}\underset{\sim}{b} + \underset{\sim}{a}) = -\underset{\sim}{b} + \underset{\sim}{a} + \frac{1}{2}\ell\underset{\sim}{b} - \ell\underset{\sim}{a}$

$\overrightarrow{CG} = (1-\ell)\underset{\sim}{a} + (\frac{1}{2}\ell - 1)\underset{\sim}{b} \cdots ①$

$\overrightarrow{CG} = \frac{1}{2}h\underset{\sim}{a} - h\underset{\sim}{b} \cdots ②$

From ① and ②, $\frac{1}{2}h = 1-\ell \cdots ③$, $-h = \frac{1}{2}\ell - 1 \cdots ④$

→ $h = 2 - 2\ell \cdots$ (3a)

Add (3a) and ④, $0 = 1 - \frac{3}{2}\ell$ → $\boxed{\ell = \frac{2}{3}}$

Sub into (3a): $h = 2 - 2(\frac{2}{3})$ → $\boxed{h = \frac{2}{3}}$

To prove the third median (line A - G - D)

$\overrightarrow{AG} = \overrightarrow{AC} + \overrightarrow{CG} = \underset{\sim}{b} + \frac{2}{3}\overrightarrow{CF}$

$= \underset{\sim}{b} + \frac{2}{3}(-\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}) = \frac{1}{3}\underset{\sim}{a} + \frac{1}{3}\underset{\sim}{b}$

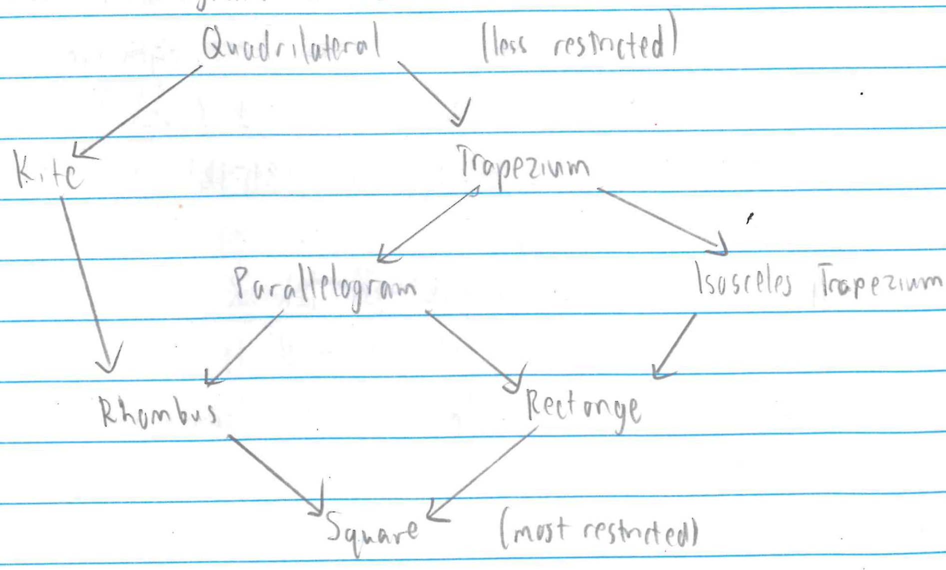
$\overrightarrow{GD} = \overrightarrow{GC} + \overrightarrow{CD} = -\frac{2}{3}(-\underset{\sim}{b} + \frac{1}{2}\underset{\sim}{a}) + \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{a})$

$= \frac{1}{6}\underset{\sim}{a} + \frac{1}{6}\underset{\sim}{b}$

$\therefore \overrightarrow{AG} = 2\overrightarrow{GD}$ → $\overrightarrow{AG} \parallel \overrightarrow{GD}$

→ AD is the third median, also divided into ratio AG:GD = 2:1 by G. (Shown)

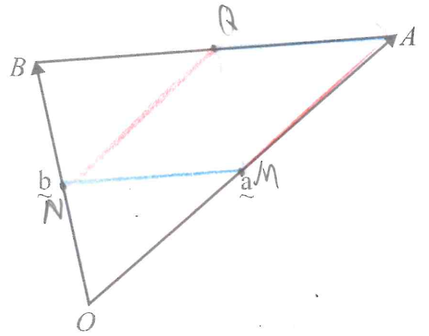
Diagram of Shapes



VCAA 2010 Exam 2 Q1

Question 1

The diagram below shows a triangle with vertices O, A and B. Let O be the origin, with vectors $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$.



a. Find the following vectors in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

i. $\overrightarrow{MA}$, where M is the midpoint of the line segment OA

$\overrightarrow{MA} = \frac{1}{2}\underset{\sim}{a}$

ii. $\overrightarrow{BA}$

$\overrightarrow{BA} = \overrightarrow{BO} + \overrightarrow{OA}$

$= \boxed{-\underset{\sim}{b} + \underset{\sim}{a}}$

iii. $\overrightarrow{AQ}$, where Q is the midpoint of the line segment AB.

$\overrightarrow{AQ} = \frac{1}{2}\overrightarrow{AB}$

$= \boxed{\frac{1}{2}(-\underset{\sim}{a} + \underset{\sim}{b})}$

1 + 1 + 1 = 3 marks

b. Let N be the midpoint of the line segment OB. Use a vector method to prove that the quadrilateral $MNQA$ is a parallelogram.

(3 marks)

- $\overrightarrow{MA} = \frac{1}{2}\underset{\sim}{a}$ (from part (a)(i))

$\overrightarrow{NQ} = \overrightarrow{NB} + \overrightarrow{BQ} = \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}\overrightarrow{BA}$ [$\overrightarrow{BQ} = \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{a})$ (from part (a)(ii)]

$= \frac{1}{2}\underset{\sim}{b} + \frac{1}{2}(-\underset{\sim}{b} + \underset{\sim}{a})$

Hence, $\overrightarrow{MA} = \overrightarrow{NQ}$. Therefore, quadrilateral $MNQA$ is a parallelogram. (parallel and equal in length)

Now consider the **particular** triangle OAB with $\overrightarrow{OA} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k}$ and $\overrightarrow{OB} = \alpha\underset{\sim}{i}$ where α, which is greater than zero, is chosen so that triangle OAB is isosceles, with $|\overrightarrow{OB}| = |\overrightarrow{OA}|$.

c. Show that $\alpha = 4$.

(1 mark)

$|\overrightarrow{OA}| = |\overrightarrow{OB}| = \sqrt{3^2 + 2^2 + (\sqrt{3})^2} = \sqrt{16} = 4$

$\therefore \boxed{\alpha = 4}$ (as req.)

(1 mark)

d. **i.** Find $\overrightarrow{OQ}$, where Q is the midpoint of the line segment AB.

$\overrightarrow{OQ} = \overrightarrow{OA} + \overrightarrow{AQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= 3\underset{\sim}{i} + 2\underset{\sim}{j} + \sqrt{3}\underset{\sim}{k} + \frac{1}{2}(\underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k})$

$= \boxed{\frac{7}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{\sqrt{3}}{2}\underset{\sim}{k}}$

ii. Use a vector method to show that $\overrightarrow{OQ}$ is perpendicular to $\overrightarrow{AB}$. (3 marks)

$\overrightarrow{AB} = \underset{\sim}{i} - 2\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k}$, $\overrightarrow{OQ} = \frac{7}{2}\underset{\sim}{i} + \underset{\sim}{j} + \frac{\sqrt{3}}{2}\underset{\sim}{k}$

$\overrightarrow{AB} \cdot \overrightarrow{OQ} = 1\left(\frac{7}{2}\right) + (-2)(1) + (-\sqrt{3})\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{7}{2} - 2 - \frac{3}{2} = 0$

Hence since $\overrightarrow{AB} \neq 0$ and $\overrightarrow{OQ} \neq 0$, and $\overrightarrow{AB} \cdot \overrightarrow{OQ} = 0$,

$\rightarrow \boxed{\overrightarrow{OQ} \perp \overrightarrow{AB} \text{ (as req.)}}$

VCAA 2014 Exam 2 Q3a (5 marks)

Let $\underset{\sim}{a} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$.

Express $\underset{\sim}{a}$ as the sum of two vector resolutes, one of which is parallel to $\underset{\sim}{b}$ and the other of which is perpendicular to $\underset{\sim}{b}$. Identify clearly the parallel vector resolute and the perpendicular vector resolute.

$\rightarrow \underset{\sim}{a} = \underset{\sim}{a}_{\parallel b} + \underset{\sim}{a}_{\perp b} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} + \underset{\sim}{a}_{\perp b}$

$\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} = \left(\frac{3 \times 2 - 2 \times 2 - 1}{2^2 + 2^2 + 1^2}\right)(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}) = \frac{1}{9}\underset{\sim}{b}$

$= \frac{1}{9}(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k})$ (parallel resolute)

$\underset{\sim}{a}_{\perp b} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel b} = 3\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k} - \left(\frac{2}{9}\underset{\sim}{i} - \frac{2}{9}\underset{\sim}{j} - \frac{1}{9}\underset{\sim}{k}\right)$

$= \frac{25}{9}\underset{\sim}{i} + \frac{20}{9}\underset{\sim}{j} + \frac{10}{9}\underset{\sim}{k}$ (perpendicular resolute)

$\therefore \underset{\sim}{a} = \boxed{\frac{1}{9}(2\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}) + \frac{5}{9}(5\underset{\sim}{i} + 4\underset{\sim}{j} + 2\underset{\sim}{k})}$

Shortest Distance example (Ex 2D)

Three points, A, B and O, are given by $A(2,1,2)$, $B(2,2,0)$ and $O(0,0,0)$.

a. Find the vector $\overrightarrow{AB}$ expressed in the form $x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$

Worked solution

$\overrightarrow{OA} = 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$

$\overrightarrow{OB} = 2\underset{\sim}{i} + 2\underset{\sim}{j}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 2\underset{\sim}{i} + 2\underset{\sim}{j} - (2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$

$= \underset{\sim}{j} - 2\underset{\sim}{k}$

1 mark

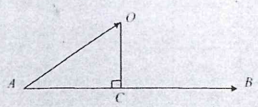
Mark allocation

- 1 mark for the correct answer.

b. A point, C, on vector $\overrightarrow{AB}$ is closest to O. Find the coordinates of point C

(3 marks)

Worked solution



Let C be the point on $\overrightarrow{AB}$ where $\overrightarrow{OC}$ is perpendicular to $\overrightarrow{AB}$.

$\overrightarrow{AC} = \frac{\overrightarrow{AO}.\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|} \times \frac{\overrightarrow{AB}}{\left|\overrightarrow{AB}\right|}$

$= \frac{1}{\left|\overrightarrow{AB}\right|^2}\left(\overrightarrow{AO}.\overrightarrow{AB}\right)\overrightarrow{AB}$

$= \left[\frac{1}{\left(\sqrt{1^2 + (-2)^2}\right)^2}(-2\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}).(\underset{\sim}{j} - 2\underset{\sim}{k})\right](\underset{\sim}{j} - 2\underset{\sim}{k})$

$= \frac{1}{5}(3)(\underset{\sim}{j} - 2\underset{\sim}{k})$

$= \frac{3}{5}\underset{\sim}{j} - \frac{6}{5}\underset{\sim}{k}$

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= 2\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k} + \frac{3}{5}\underset{\sim}{j} - \frac{6}{5}\underset{\sim}{k}$

$= 2\underset{\sim}{i} + \frac{8}{5}\underset{\sim}{j} + \frac{4}{5}\underset{\sim}{k}$

$\Rightarrow C$ is $\left(2, \frac{8}{5}, \frac{4}{5}\right)$

VCAA 2012 Exam 2 Q5 (10 marks) (Ch 4)

Let $u = 6-2i$ and $w = 1+3i$ where $u, w \in C$.

(a) Given that $z_1 = \frac{(u+w)\bar{u}}{iw}$, show that $|z_1| = 10\sqrt{2}$. (1 mark)

$\rightarrow u+w = 7+i$, $\bar{u} = 6+2i$

$$z_1 = \frac{(7+i)(6+2i)}{i(1+3i)} = \frac{42+6i+14i-2}{-3+i} = \frac{40+20i}{-3+i}$$

$$= \frac{20(2+i)(-3-i)}{(-3+i)(-3-i)} = \frac{20(-6-3i-2i+1)}{9+1} = \frac{20}{10}(-5-5i)$$

$z_1 = -10-10i$

$|z_1| = \sqrt{(-10)^2+(-10)^2} = \boxed{10\sqrt{2}}$ (Shown)

(b) The complex number z_1 can be expressed in polar form as

$z_1 = 200^{1/2} \text{cis}\left(-\frac{3\pi}{4}\right)$

Find all distinct complex numbers z such that $z^3 = z_1$ (3 marks)

Give answers in form $a^{1/n} \text{cis}\left(\frac{b\pi}{c}\right)$, where a, b, c and n are integers.

$\rightarrow z^3 = 200^{1/2} \text{cis}\left(-\frac{3\pi}{4} + 2k\pi\right)$, $k \in Z$

$z = \left(200^{1/2}\right)^{1/3} \text{cis}\left(-\frac{\pi}{4} + \frac{2k\pi}{3}\right)$, $k \in Z$

If $k=0$, $z = 200^{1/6} \text{cis}\left(-\frac{\pi}{4}\right)$

If $k=1$, $z = 200^{1/6}\left(-\frac{3\pi}{12} + \frac{8\pi}{12}\right) = 200^{1/6} \text{cis}\left(\frac{5\pi}{12}\right)$

If $k=2$, $z = 200^{1/6}\left(-\frac{3\pi}{12} + \frac{16\pi}{12}\right) = 200^{1/6} \text{cis}\left(\frac{13\pi}{12}\right) = 200^{1/6} \text{cis}\left(-\frac{11\pi}{12}\right)$

$\therefore \boxed{z = 200^{1/6} \text{cis}\left(-\frac{\pi}{4}\right),\ z = 200^{1/6} \text{cis}\left(\frac{5\pi}{12}\right),\ z = 200^{1/6} \text{cis}\left(-\frac{11\pi}{12}\right)}$

Let argument of u be given by $\text{Arg}(u) = -\alpha$

(c) By expressing iw in polar form in terms of α, show that

$\frac{\bar{u}}{iw} = 2\text{cis}(2\alpha - \pi)$ (3 marks)

$\rightarrow u = |z_1| \arg(z_1) = \left(\sqrt{6^2+(-2)^2}\right) \arg(z_1)$

$u = 2\sqrt{10} \text{cis}(-\alpha) \rightarrow \bar{u} = 2\sqrt{10} \text{cis}(\alpha)$

$\rightarrow iw = i(1+3i) = -3+i = -\frac{1}{2}(6-i)$ (N.B. $\text{cis}(-\alpha) = \text{cis}(\pi-\alpha)$)

$iw = -\frac{1}{2}u = -\frac{1}{2}\left(2\sqrt{10} \text{cis}(\alpha)\right) = -\sqrt{10} \text{cis}(\alpha) \rightarrow iw = \sqrt{10} \text{cis}(\pi-\alpha)$

$\rightarrow \frac{\bar{u}}{iw} = \frac{2\sqrt{10} \text{cis}(\alpha)}{\sqrt{10} \text{cis}(\pi-\alpha)} = \boxed{2\text{cis}(2\alpha-\pi)}$ (Shown)

(d) Use the relation given in part (a) to find $\text{Arg}(u+w)$ in terms of α (3 marks)

$\rightarrow$ From (a), $z_1 = \frac{\bar{u}}{iw}(u+w)$, From (b), $z_1 = 200^{1/2} \text{cis}\left(-\frac{3\pi}{4}\right)$

$\therefore \text{Arg}(z_1) = \text{Arg}\left(\frac{\bar{u}}{iw}\right) + \text{Arg}(u+w)$

From (b), $-\frac{3\pi}{4} = 2\alpha - \pi + \text{Arg}(u+w)$

$\therefore \text{Arg}(u+w) = -\frac{3\pi}{4} - (2\alpha - \pi) = \boxed{\frac{\pi}{4} - 2\alpha}$

Complex Numbers Challenge |Ex 4I1|

In the complex plane, C is the circle with equation $|z+5-i| = \sqrt{2}$. (2 marks)

a. Show that the cartesian equation of C is given by $(x+5)^2 + (y-1)^2 = 2$.

a. $|z+5-i| = \sqrt{2}$ where $z = x + yi$ — A1

$|x+5+(y-1)i| = \sqrt{2}$

$\sqrt{(x+5)^2+(y-1)^2} = \sqrt{2}$

Squaring both sides we obtain $(x+5)^2+(y-1)^2 = 2$. — A1

In the complex plane, L is the half-line with equation $\operatorname{Arg}(z+2i) = \dfrac{3\pi}{4}$. (2 marks)

b. Show that the cartesian equation of L is given by $y = -x-2,\ x<0$.

b. Method 1:

$\operatorname{Arg}(z) = \frac{3\pi}{4}$ is the half-line emanating from O, but not including O, which makes an angle of $\frac{3\pi}{4}$ with the positive $\operatorname{Re}(z)$ direction.

This half-line has a cartesian equation given by $y = -x$. — A1

$\operatorname{Arg}(z+2i) = \frac{3\pi}{4}$ is the half-line of $\operatorname{Arg}(z) = \frac{3\pi}{4}$ translated -2 units in the $\operatorname{Im}(z)$ direction.

Hence L has a cartesian equation given by $y = -x-2,\ x<0$. — A1

Use alternatively Method 2:

Let $z = x + yi$

So $z + 2i = x + (y+2)i$.

$\tan\left(\frac{3\pi}{4}\right) = \frac{y+2}{x},\ y > -2$ and $x < 0$. — A1

As $\tan\left(\frac{3\pi}{4}\right) = -1$, we obtain $\frac{y+2}{x} = -1,\ x<0$.

Hence L has a cartesian equation given by $y = -x-2,\ x<0$. — A1

In the complex plane, point B has coordinates $(-4, 2)$.

c. Verify that point B lies on L **and** also lies on C.

c. Point B has coordinates $(-4, 2)$.

Substituting $x=-4$ into $y=-x-2,\ x<0$ gives $y=2$ and substituting $x=-4$ and $y=2$ into $(x+5)^2+(y-1)^2$ we obtain $(-4+5)^2+(2-1)^2 = 2$. — A1

Hence point B lies on L and also lies on C.

d. Hence, or otherwise, show that L touches C. (3 marks)

d. Method 1:

$\frac{d}{dx}((x+5)^2) + \frac{d}{dx}((y-1)^2) = \frac{d}{dx}(2)$

$2(x+5) + 2(y-1)\frac{dy}{dx} = 0$ — M1

$\frac{dy}{dx} = \frac{-(x+5)}{y-1}$ — A1

At $(-4, 2)$, the gradient of both C and L is -1. So L touches C. — A1

Use alternatively Method 2:

The gradient of the line (radius) joining $(-5, 1)$ and $(-4, 2)$ is $m = \frac{2-1}{-4-(-5)} = 1$. — A1

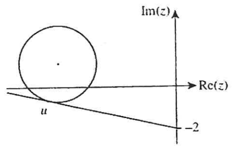
L has a gradient of -1.

The product of the two gradients is -1. — A1

Hence the radius is perpendicular to L and therefore is a tangent. So L touches C. — A1

The complex number u lies on C and is such that $\operatorname{Arg}(u+2i)$ has its maximum value.

e. Find u in exact cartesian form and find the exact maximum value of $\operatorname{Arg}(u+2i)$, expressing your answer in the form $\pi - \tan^{-1}\left(\frac{p}{q}\right)$ where p and q are positive integers. (6 marks)



Method 1:

Let the cartesian equation of the movable half-line be $y = mx - 2,\ x<0$.

Let $u = x + yi$.

Substituting $y = mx-2$ into $(x+5)^2+(y-1)^2 = 2$ gives $(m^2+1)x^2 + (10-6m)x + 34 = 2$.

Solving $(m^2+1)x^2+(10-6m)x+34 = 2$ for x gives $x = \dfrac{3m-5\pm\sqrt{-23m^2-30m-7}}{m^2+1}$. — M1

Solving $-23m^2-30m-7 = 0$ for m gives $m=-1$ or $m = -\frac{7}{23}$. Reject $m=-1$. — A1

Solving $(x+5)^2+(y-1)^2 = 2$ and $y = -\frac{7}{23}x - 2$ for x and y gives $x = -\frac{92}{17}$ and $y = -\frac{6}{17}$. — M1 A1

Hence $u = -\frac{92}{17} - \frac{6}{17}i$ and so $u + 2i = -\frac{92}{17} + \frac{28}{17}i$. — A1

$\operatorname{Arg}\left(-\frac{92}{17}+\frac{28}{17}i\right) = \pi - \tan^{-1}\left(\frac{7}{23}\right)$ — A1

Use alternatively Method 2:

Let the cartesian equation of the movable half-line be $y = mx-2,\ x<0$.

Let $u = x + yi$.

Substituting $y = mx - 2$ into $(x+5)^2+(y-1)^2 = 2$ gives $(m^2+1)x^2+(10-6m)x+32 = 0$.

$\Delta = (10-6m)^2 - 4\times 32\times(m^2+1)$ — M1

Solving $(10-6m)^2 - 4\times 32\times(m^2+1) = 0$ (or equivalent) for m gives $m=-1$ or $m = -\frac{7}{23}$. — A1

Reject $m = -1$.

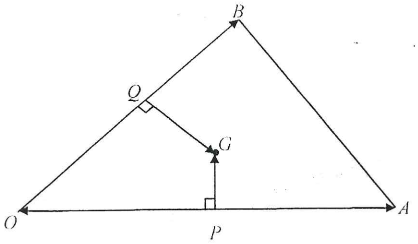
Solving $(x+5)^2+(y-1)^2 = 2$ and $y = -\frac{7}{23}x - 2$ for x and y gives $x = -\frac{92}{17}$ and $y = -\frac{6}{17}$. — M1 A1

Hence $u = -\frac{92}{17} - \frac{6}{17}i$ and so $u + 2i = -\frac{92}{17}+\frac{28}{17}i$. — A1

$\operatorname{Arg}\left(-\frac{92}{17}+\frac{28}{17}i\right) = \pi - \tan^{-1}\left(\frac{7}{23}\right)$ — A1

(Ex 2F)

a. In the triangle OAB, P and Q are the midpoints of OA and OB respectively and G is a point inside the triangle. The vectors $\overrightarrow{PG}$ and $\overrightarrow{QG}$ are perpendicular to the sides OA and OB respectively. Let $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$ and $\overrightarrow{OG} = g$.



i. Express $\overrightarrow{PG}$ and $\overrightarrow{QG}$ in terms of a, b and g and hence show that $g.a = \frac{1}{2}|a|^2$ and $g.b = \frac{1}{2}|b|^2$.

3 marks

(i)

$\overrightarrow{PG} = \overrightarrow{PO} + \overrightarrow{OG}$

Since P is the midpoint of OA.

$= -\frac{1}{2}\overrightarrow{OA} + \overrightarrow{OG}$

$= g - \frac{1}{2}a$

$\overrightarrow{QG} = \overrightarrow{QO} + \overrightarrow{OG}$

Since Q is the midpoint of OB

$= -\frac{1}{2}\overrightarrow{OB} + \overrightarrow{OG}$

$= g - \frac{1}{2}b$ A1

Since $\overrightarrow{PG}$ is perpendicular to $\overrightarrow{OA}$. $\overrightarrow{PG}.\overrightarrow{OA} = 0$

$\left(g - \frac{1}{2}a\right).a = 0$

$g.a - \frac{1}{2}a.a = 0$ M1

$g.a = \frac{1}{2}|a|^2$

Similarly since $\overrightarrow{QG}$ is perpendicular to $\overrightarrow{OB}$. $\overrightarrow{QG}.\overrightarrow{OB} = 0$

$\left(g - \frac{1}{2}b\right).b = 0$

$g.b - \frac{1}{2}b.b = 0$ A1

$g.b = \frac{1}{2}|b|^2$

ii. Let R be the midpoint of AB, show that RG is perpendicular to AB. (4 marks)

(ii)

$\overrightarrow{RG} = \overrightarrow{RA} + \overrightarrow{AP} + \overrightarrow{PG}$

$= \frac{1}{2}\overrightarrow{BA} + \overrightarrow{AP} + \overrightarrow{PG}$ Since R is the midpoint of AB M1

$= \frac{1}{2}(a - b) - \frac{1}{2}a + \left(g - \frac{1}{2}a\right)$

$= g - \frac{1}{2}(a + b)$ A1

Consider $\overrightarrow{RG}.\overrightarrow{AB} = \left(g - \frac{1}{2}(a + b)\right).(b - a)$

$= g.b - \frac{1}{2}(a + b).(b - a) - g.a$

$= g.b - \frac{1}{2}(a.b + b.b - a.a - b.a) - g.a$ from i. A1

$= \frac{1}{2}|b|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a|^2 - \frac{1}{2}|a|^2 = 0$

so therefore $\overrightarrow{RG}$ is perpendicular to $\overrightarrow{AB}$ A1

VCAA NH 2018 Exam 1 Q2

Question 2 (3 marks)

Let $a = 3i - 2j + mk$ and $b = 2i - j + 3k$, where $m \in R$.

Find the value(s) of m such that the magnitude of the vector resolute of a parallel to b is equal to $\sqrt{14}$.

Scalar resolute of a onto b

$= a.\hat{b}$

$= \frac{a.b}{|b|}$

$= \frac{6+2+3m}{\sqrt{14}}$

$= \frac{8+3m}{\sqrt{14}}$

So $\left|\frac{8+3m}{\sqrt{14}}\right| = \sqrt{14}$

$\rightarrow$ $8+3m = \pm 14$

If $8+3m = 14$

$\rightarrow$ $3m = 6$

$\rightarrow$ $\boxed{m = 2}$

If $8+3m = -14$

$\rightarrow$ $3m = -22$

$\rightarrow$ $\boxed{m = -\frac{22}{3}}$

VCAA NH 2017 Exam 2 Q2 (Ch 4)

One root of a quadratic equation with real coefficients is $\sqrt{3}+i$.

a. **i.** Write down the other root of the quadratic equation. 1 mark

$\sqrt{3}-i$

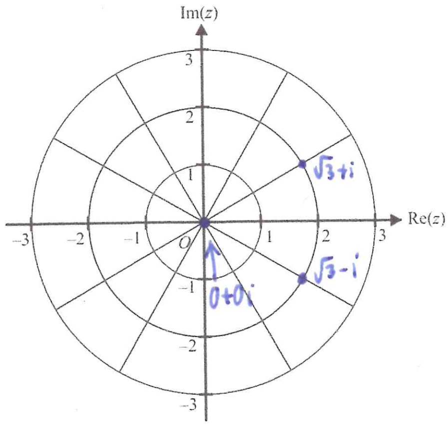
ii. Hence determine the quadratic equation, writing it in the form $z^2 + bz + c = 0$. 2 marks

$(z-(\sqrt{3}+i))(z-(\sqrt{3}-i))=0$

$(z-\sqrt{3})^2 + i^2 = 0$

$z^2 - 2\sqrt{3}z + 4 = 0$

b. Plot and label the roots of $z^3 - 2\sqrt{3}z^2 + 4z = 0$ on the Argand diagram below. 3 marks



c. Find the equation of the line that is the perpendicular bisector of the line segment joining the origin and the point $\sqrt{3}+i$. Express your answer in the form $y = mx + c$. 2 marks

Midpoint: $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ Gradient (tangent): $\frac{1}{\sqrt{3}}$

Gradient of normal $= -\sqrt{3}$

$y - \frac{1}{2} = -\sqrt{3}(x - \sqrt{3}/2) \rightarrow \boxed{y = -\sqrt{3}x + 2}$

d. The three roots plotted in **part b.** lie on a circle.

Find the equation of this circle, expressing it in the form $|z-\alpha| = \beta$, where $\alpha, \beta \in R$. 3 marks

Sub $z = \sqrt{3}+i$, $|\sqrt{3}+i-\alpha| = \beta$

$\sqrt{(\sqrt{3}-\alpha)^2 + 1^2} = \beta$ ... ①

Sub $z = 0+0i$, $|-\alpha| = \beta \rightarrow \alpha = \beta$ ... ②

Solve ① and ②, $\alpha = \frac{2\sqrt{3}}{3}$, $\beta = \frac{2\sqrt{3}}{3}$

simultaneously $\boxed{\left|z - \frac{2\sqrt{3}}{3}\right| = \frac{2\sqrt{3}}{3}}$

(F) Find the area of the major segment bounded by the line $\mathrm{Re}(z) = \sqrt{3}$ and the major arc of the circle given by $|z| = 2$.

[Extra Question = 2 marks]

Area $= \frac{1}{2} \times 2^2 \times \left(\frac{5\pi}{3} - \sin\left(\frac{5\pi}{3}\right)\right)$

$= \left(\frac{10\pi}{3} + \sqrt{3}\right)$ units^2

N.B. major, not minor arc

(use $\theta = \frac{5\pi}{3}$, not $\theta = \frac{\pi}{3}$)

→ Read Q specifically!

VCAA MM 2018 Exam 2 Q2 (ch 4)

In the complex plane, L is the line given by $|z+1| = \left|z + \frac{1}{2} - \frac{\sqrt{3}}{2}i\right|$.

a. Show that the cartesian equation of L is given by $y = -\frac{1}{\sqrt{3}}x$. 2 marks

$|x+yi+1| = |x+yi+\frac{1}{2}-\frac{\sqrt{3}}{2}i|$

$|x+1+yi| = |x+\frac{1}{2}+(y-\frac{\sqrt{3}}{2})i|$

$\sqrt{(x+1)^2+y^2} = \sqrt{(x+\frac{1}{2})^2+(y-\frac{\sqrt{3}}{2})^2}$

$x^2+2x+1+y^2 = x^2+x+\frac{1}{4}+y^2-\sqrt{3}y+\frac{3}{4}$

$x = -\sqrt{3}y \to \boxed{y = \frac{-1}{\sqrt{3}}x}$ (shown)

b. Find the point(s) of intersection of L and the graph of the relation $z\bar{z} = 4$ in cartesian form. 2 marks

$(x+yi)(x-yi) = 4 \to x^2+y^2 = 4$ --- ①

$y = -\frac{1}{\sqrt{3}}x$ --- ②

Sub ② into ①, $x^2 + (-\frac{1}{\sqrt{3}}x)^2 = 4$

$\frac{4}{3}x^2 = 4 \to x^2 = 3$

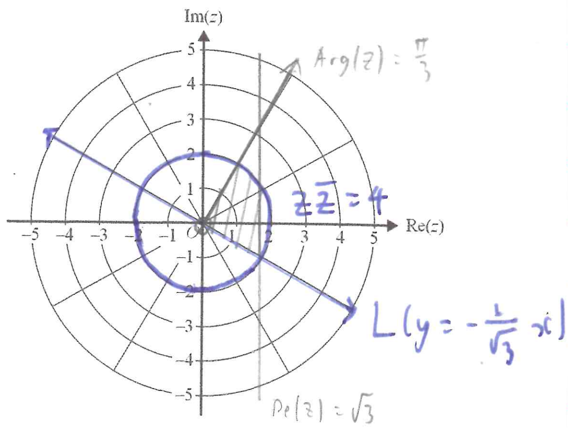
$x = \pm\sqrt{3}$

When $x = -\sqrt{3}, y = 1$. When $x = \sqrt{3}, y = -1$.

$\boxed{(-\sqrt{3}, 1), (\sqrt{3}, -1)}$

N.B. Answer in coordinates, not complex form a+bi

c. Sketch L and the graph of the relation $z\bar{z} = 4$ on the Argand diagram below. 2 marks



The part of the line L in the fourth quadrant can be expressed in the form $\text{Arg}(z) = \alpha$.

d. State the value of α. 1 mark

$\alpha = -\frac{\pi}{6}$

e. Find the area enclosed by L and the graphs of the relations $z\bar{z} = 4$, $\text{Arg}(z) = \frac{\pi}{3}$ and $\text{Re}(z) = \sqrt{3}$. 2 marks

$\text{Area} = \frac{1}{4}\pi(2)^2 - \left[\frac{1}{2}(2)^2\left(\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\right)\right]$

$= \pi - \left(\frac{2\pi}{3} - \sqrt{3}\right)$

$= \boxed{\left(\frac{\pi}{3} + \sqrt{3}\right)}$ units^2

(f) The straight line L can be written in the form $z = h\bar{z}$, where $h \in C$.

Find h in the form $r\text{cis}(\theta)$, where θ is the principal argument of h. (2 marks)

$\to x+yi = h(x-yi)$

$h = \frac{x+yi}{x-yi} = \frac{x^2-y^2}{x^2+y^2} + \frac{2xy}{x^2+y^2}i$ ... ①

Sub $y = -\frac{1}{\sqrt{3}}x$ into ①,

$h = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$\therefore \boxed{h = \text{cis}(-\frac{\pi}{3})}$

VCAA 2014 Exam 2 Q2

Question 2 (13 marks)

Consider the complex number $z_1 = \sqrt{3} - 3i$.

a. **i.** Express z_1 in polar form. 2 marks

$|z_1| = \sqrt{(\sqrt{3})^2 + (-3)^2} = 2\sqrt{3}$

$\tan(\theta) = \frac{3}{\sqrt{3}} = \sqrt{3} \rightarrow \theta = \frac{\pi}{3} \rightarrow Arg(z_1) = -\frac{\pi}{3}$

$\therefore \boxed{z_1 = 2\sqrt{3}\, cis\left(-\frac{\pi}{3}\right)}$

$\sqrt{3}$, θ, 3

ii. Find $\text{Arg}(z_1^4)$. 1 mark

$Arg(z_1^4) = -\frac{4\pi}{3} \rightarrow \boxed{\frac{2\pi}{3}}$

iii. Given that $z_1 = \sqrt{3} - 3i$ is one root of the equation $z^3 + 24\sqrt{3} = 0$, find the other two roots, expressing your answers in cartesian form. 2 marks

$P(z) = z^3 + 24\sqrt{3} = (z - (\sqrt{3} - 3i))(z - (\sqrt{3} + 3i)).(z - \alpha)$

$= ((z - \sqrt{3})^2 + 9)(z - \alpha) = (z^2 - 2\sqrt{3}z + 12)(z - \alpha)$

Equating coefficients $24\sqrt{3} = -12\alpha \rightarrow \alpha = -2\sqrt{3}$

Other two roots: $\boxed{z = \sqrt{3} + 3i, \quad z = -2\sqrt{3}}$

b. **i.** Find the value of $(z_1 + 2i)(\bar{z}_1 - 2i)$, where $z_1 = \sqrt{3} - 3i$. 1 mark

4

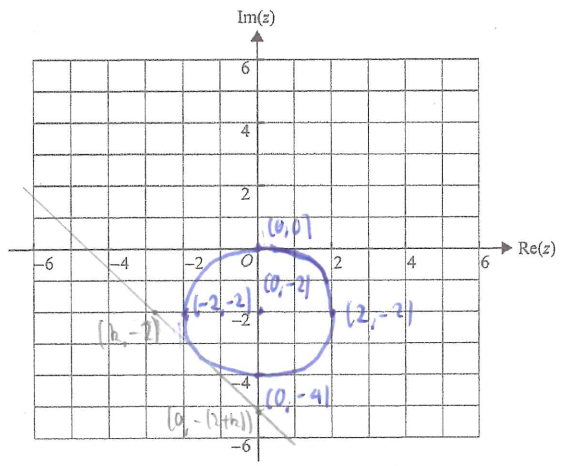
ii. Show that the relation $(z + 2i)(\bar{z} - 2i) = 4$ can be expressed in cartesian form as $x^2 + (y+2)^2 = 4$. 2 marks

$(x + yi + 2i)(x - yi - 2i) = 4 \rightarrow (x + (y+2)i)(x - (y+2)i) = 4$

$x^2 - (y+2)^2 i^2 = 4$

$\boxed{x^2 + (y+2)^2 = 4}$

iii. Sketch $\{z : (z + 2i)(\bar{z} - 2i) = 4\}$ on the axes below. 2 marks



c. The line joining the points corresponding to $k - 2i$ and $-(2 + k)i$, where $k < 0$, is tangent to the curve given by $\{z : (z + 2i)(\bar{z} - 2i) = 4\}$.

Find the value of k. 3 marks

Strategy:–

1. Find equation of line joining $k - 2i$ and $-(2+k)i$
2. Set up an equation for the intersection of the line and the circle.
3. Solve the equation above for only one solution point (tangent)

So the two points are $(k, -2)$ and $(0, -(2+k))$

$m_{line} = \frac{-2 - (-(2+k))}{k - 0}$

$= \frac{-k}{k} = -1$

$\rightarrow$ Eq^n of line passing through $(k, -2)$ is

$y - y_1 = m(x - x_1)$

ie $y - (-2) = -1(x - k)$

$\rightarrow y = -x + (k - 2)$ — ①

Also, $x^2 + (y+2)^2 = 4$ — ②

Sub for y from ① into ②:–

② $\rightarrow x^2 + (-x + k \cancel{-2+2})^2 = 4$

$\rightarrow x^2 + (x - k)^2 = 4$

$\rightarrow x^2 + x^2 - 2kx + k^2 = 4$

$\rightarrow 2x^2 - (2k)x + (k^2 - 4) = 0$

$\Delta = (-2k)^2 - 4 \times 2 \times (k^2 - 4) = 0$

$\rightarrow 4k^2 - 8(k^2 - 4) = 0$

$\rightarrow -4k^2 + 32 = 0$

$\rightarrow k^2 = 8 \rightarrow \boxed{k = -2\sqrt{2}}$ $(k < 0)$