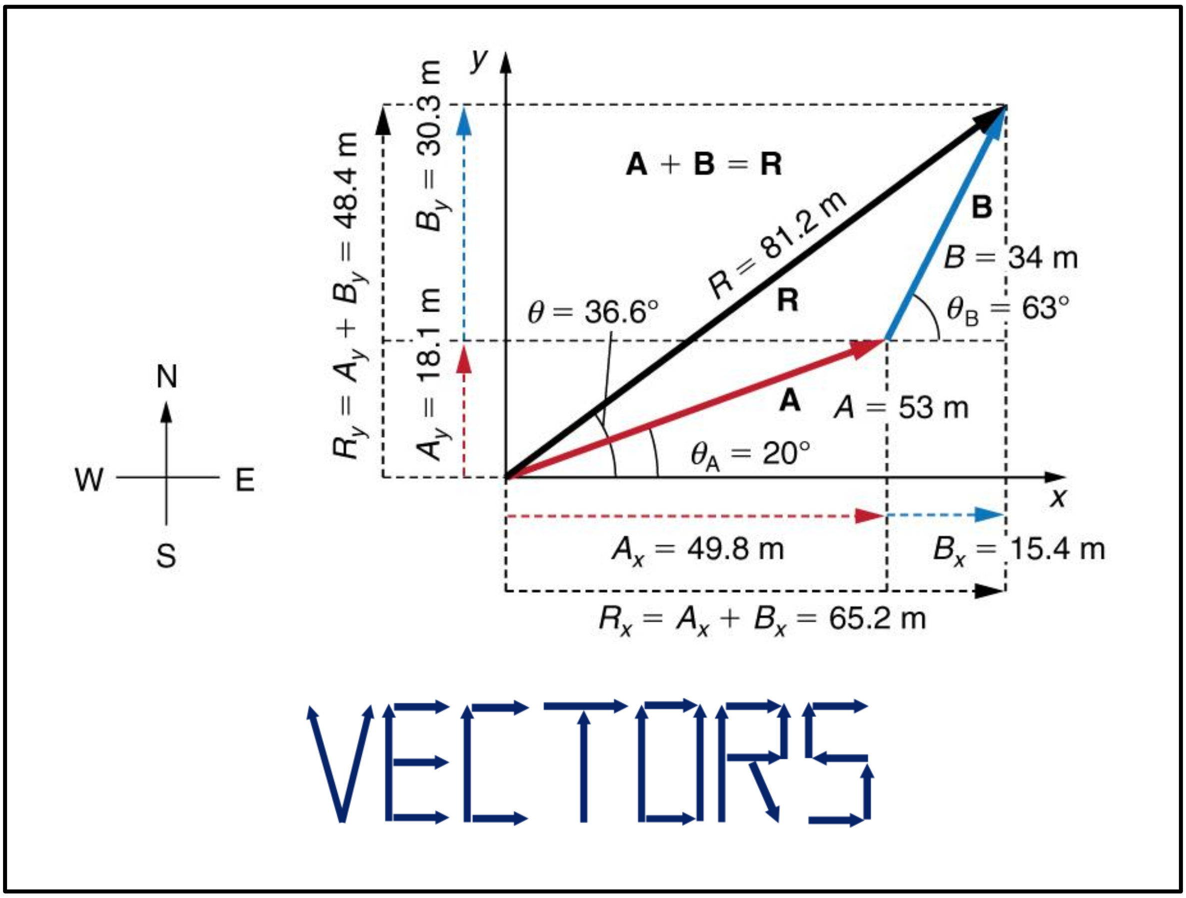
UNIT 3 SPECIALIST MATHS

VECTORS & VECTOR CALCULUS



WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

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Chapter 2 Vectors

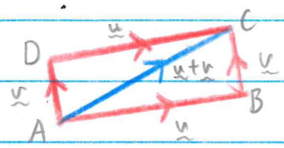
Exercise 2A Introduction to vectors

→ Vectors - measured using both magnitude and direction
- e.g. force, displacement, velocity, acceleration

→ Representation → directed line segments / column vectors
- Magnitude → length of line
- Direction → arrow head
- N.B. tilde underneath! → $\underset{\sim}{u}$, $\underset{\sim}{v}$

→ Magnitude of vectors → length of line (Pythagoras' theorem)
- $|\overrightarrow{AB}| = |\underset{\sim}{u}| = \sqrt{x^2 + y^2}$

→ Addition of vectors
- $\underset{\sim}{u} + \underset{\sim}{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$
- $\underset{\sim}{v} + \underset{\sim}{u} = \overrightarrow{AD} + \overrightarrow{DC} = \overrightarrow{AC}$

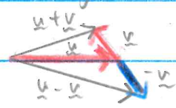


- N.B. 'distance travelled' / 'length journey' not relevant
  line of sight 'displacement' from start to finish relevant

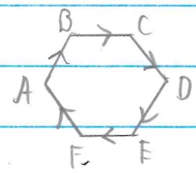
→ Scalar multiplication - shortens / lengthens vector with direction constant

→ Zero vector - no length + no direction

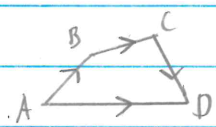
→ Subtraction of vectors - add negative of vector being subtracted
- $\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{u} + (-\underset{\sim}{v})$



→ Polygon of vectors
- e.g.1. polygon ABCDEF
  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = 0$



- e.g.2. polygon ABCD
  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$



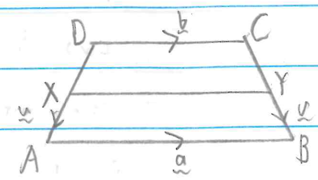
→ Parallel vectors - have identical directions
- relative proportions of x, y and z components preserved
- $\underset{\sim}{u} = k\underset{\sim}{v}$ if $k \in R \setminus \{0\}$

→ Position vectors - position in space of known point relative to origin O
- denoted as $\overrightarrow{OA}$

→ 3D vectors → $a\underset{\sim}{i} + b\underset{\sim}{j} + c\underset{\sim}{k}$

→ Linear combinations of non-parallel vectors

- e.g.1. ABCD is a trapezium with AB parallel to DC. X and Y are midpoints of AD and BC respectively.



(a) Express $\overrightarrow{XY}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where $\underset{\sim}{a} = \overrightarrow{AB}$ and $\underset{\sim}{b} = \overrightarrow{DC}$.

(b) Show that XY is parallel to AB.

(a) Let $\overrightarrow{DA} = \underset{\sim}{u}$ and $\overrightarrow{CB} = \underset{\sim}{v}$

$\overrightarrow{XY} = \overrightarrow{XA} + \overrightarrow{AB} + \overrightarrow{BY} = \frac{1}{2}\underset{\sim}{u} + \underset{\sim}{a} - \frac{1}{2}\underset{\sim}{v} \cdots ①$

$\overrightarrow{XY} = \overrightarrow{XD} + \overrightarrow{DC} + \overrightarrow{CY} = -\frac{1}{2}\underset{\sim}{u} + \underset{\sim}{b} + \frac{1}{2}\underset{\sim}{v} \cdots ②$

① + ②: $2\overrightarrow{XY} = \underset{\sim}{a} + \underset{\sim}{b}$

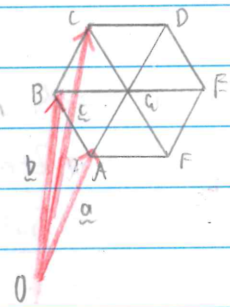
$\therefore \boxed{\overrightarrow{XY} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})}$

(b) $\overrightarrow{XY} = \frac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b})$

But $\underset{\sim}{b} = k\underset{\sim}{a}$ ($\underset{\sim}{b} \parallel \underset{\sim}{a}$), k is a constant

Hence $\boxed{\overrightarrow{XY} \parallel \underset{\sim}{a}}$ (as required)

- e.g.2. ABCDEF is a regular hexagon with centre G. The position vectors of A, B and C relative to an origin O, are $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively.



(a) Express $\overrightarrow{OG}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$

(b) Express $\overrightarrow{CD}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

(a) $\overrightarrow{OG} = \overrightarrow{OB} + \overrightarrow{BG}$

$= \underset{\sim}{b} + \overrightarrow{BC} + \overrightarrow{CG}$

$\overrightarrow{BC} = \overrightarrow{BO} + \overrightarrow{OC} = -\underset{\sim}{b} + \underset{\sim}{c}$

$\overrightarrow{CG} = \overrightarrow{BA} = -\underset{\sim}{b} + \underset{\sim}{a}$

So, $\overrightarrow{OG} = \underset{\sim}{b} - \underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{b} + \underset{\sim}{a}$

$\therefore = \boxed{\underset{\sim}{a} - \underset{\sim}{b} + \underset{\sim}{c}}$

(b) $\overrightarrow{CD} = \overrightarrow{BG} = \overrightarrow{BC} + \overrightarrow{CG}$

$= -\underset{\sim}{b} + \underset{\sim}{c} - \underset{\sim}{b} + \underset{\sim}{a}$

$\therefore = \boxed{\underset{\sim}{a} - 2\underset{\sim}{b} + \underset{\sim}{c}}$

→ Linear dependence and independence

· Linearly dependent → at least one vector from a set expressed as linear combination of other vectors in set (2 or 3)

· e.g.1. $\underset{\sim}{u} = 10\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$; $\underset{\sim}{v} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$, $\underset{\sim}{w} = 4\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}$

If linear dependent, $\begin{bmatrix} 10 \\ 3 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ -1 \\ -2 \end{bmatrix}$

Solving for α and β: $-\alpha + 4\beta = 10 \cdots ①$

$3\alpha - \beta = 3 \cdots ②$

$4\alpha - 2\beta = 2 \cdots ③$

①×3: $-3\alpha + 12\beta = 30 \cdots$ 1a

1a + ②: $11\beta = 33 \to \beta = 3$

Sub into ②, $3\alpha - 3 = 3 \to \alpha = 2$

Sub $\alpha = 2$, $\beta = 3$ into ③,

$LHS = 4\alpha - 2\beta = 4(2) - 2(3)$

$= 2 = RHS$ (consistent!)

→ $\boxed{\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \text{ are linear dependent.}}$

· e.g.2. $\underset{\sim}{u} = 10\underset{\sim}{i} + 3\underset{\sim}{j} + 2\underset{\sim}{k}$, $\underset{\sim}{v} = -\underset{\sim}{i} + 3\underset{\sim}{j} + 4\underset{\sim}{k}$ $\underset{\sim}{w} = 4\underset{\sim}{i} - \underset{\sim}{j} - 8\underset{\sim}{k}$

If linear dependent, $\begin{bmatrix} 10 \\ 3 \\ 2 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ -1 \\ -8 \end{bmatrix}$

Solving for α and β: $-\alpha + 4\beta = 10 \cdots ①$

$3\alpha - \beta = 3 \cdots ②$

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Sub into ②, $3\alpha - 3 = 3 \to \alpha = 2$

Sub $\alpha = 2$, $\beta = 3$ into ③,

$LHS = 4\alpha - 8\beta = 4(2) - 8(3)$

$= 8 - 24 = -16$

But ③ $RHS = 2 \to LHS \neq RHS$ (inconsistent!)

→ $\boxed{\underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w} \text{ are linear independent.}}$

· Alternate solution: $\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = 0$, if $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ linearly dependent e.g. 3. Consider vectors $\underset{\sim}{a} = -3\underset{\sim}{i} + 2\underset{\sim}{j} + 3\underset{\sim}{k}$, $\underset{\sim}{b} = -4\underset{\sim}{i} - 4\underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{c} = m\underset{\sim}{i} + n\underset{\sim}{k}$ where m and n are non-zero, real constants.

Find m in terms of n if $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are linearly dependent vectors.

For linear dependence, $\underset{\sim}{a} = \alpha\underset{\sim}{b} + \beta\underset{\sim}{c}$ $(\alpha, \beta \neq 0)$

$$\begin{bmatrix} -3 \\ 2 \\ 3 \end{bmatrix} = \alpha \begin{bmatrix} -4 \\ -4 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} m \\ 0 \\ n \end{bmatrix}$$

$-4\alpha + \beta m = -3 \cdots ①$

$-4\alpha = 2 \cdots ②$

$2\alpha + \beta n = 3 \cdots ③$

From ②, $\alpha = -\frac{1}{2}$

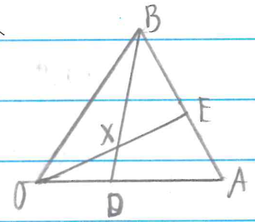
Sub into ①, $2 + m\beta = -3 \rightarrow m\beta = -5 \cdots$ (1a)

Sub into ③, $-1 + n\beta = 3 \rightarrow n\beta = 4 \cdots$ (3a)

From (1a), $\beta = -\frac{5}{m}$

Sub into (3a), $n\left(-\frac{5}{m}\right) = 4 \rightarrow \frac{n}{m} = -\frac{4}{5} \rightarrow -4m = 5n \rightarrow \boxed{m = -\frac{5}{4}n}$

e.g. 4. Points A and B have position vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ respectively, relative to an origin O.



The point D is such that $\overrightarrow{OD} = k\overrightarrow{OA}$ and point E is such that $\overrightarrow{AE} = \ell\overrightarrow{AB}$. The line segments BD and OE intersect at X.

Assume that $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE}$ and $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB}$.

(a) Express $\overrightarrow{XB}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and k.

(b) Express $\overrightarrow{OX}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and ℓ.

(c) Express $\overrightarrow{XB}$ in terms of $\underset{\sim}{a}$, $\underset{\sim}{b}$ and ℓ.

(d) Find k and ℓ.

(a) $\overrightarrow{XB} = \frac{4}{5}\overrightarrow{DB} = \frac{4}{5}(-\overrightarrow{OD} + \overrightarrow{OB})$

$= \frac{4}{5}(-k\overrightarrow{OA} + \overrightarrow{OB})$

$= \frac{4}{5}(-k\underset{\sim}{a} + \underset{\sim}{b})$

$\therefore = -\frac{4}{5}k\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b}$

(b) $\overrightarrow{OX} = \frac{2}{5}\overrightarrow{OE} = \frac{2}{5}(\overrightarrow{OA} + \overrightarrow{AE})$

$= \frac{2}{5}(\overrightarrow{OA} + \ell\overrightarrow{AB})$

$= \frac{2}{5}(\underset{\sim}{a} + \ell(\underset{\sim}{b} - \underset{\sim}{a}))$

$\therefore = \frac{2}{5}(1-\ell)\underset{\sim}{a} + \frac{2\ell}{5}\underset{\sim}{b}$

(c) $\overrightarrow{XB} = \overrightarrow{XO} + \overrightarrow{OB}$

$= -\frac{2}{5}(1-\ell)\underset{\sim}{a} - \frac{2\ell}{5}\underset{\sim}{b} + \underset{\sim}{b}$

$\therefore = \frac{2}{5}(\ell-1)\underset{\sim}{a} + \left(1-\frac{2\ell}{5}\right)\underset{\sim}{b}$

(d) From parts (a) and (c), $-\frac{4h}{5}\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b} = \frac{2}{5}(\ell-1)\underset{\sim}{a} + \left(1-\frac{2\ell}{5}\right)\underset{\sim}{b}$

Equating coefficients, $-\frac{4h}{5} = \frac{2}{5}(\ell-1) \cdots ①$, $\frac{4}{5} = 1 - \frac{2\ell}{5} \cdots ②$

From ②, $\frac{2\ell}{5} = \frac{1}{5} \rightarrow \boxed{\ell = \frac{1}{2}}$

Sub into ①, $-\frac{4h}{5} = \frac{2}{5}\left(\frac{1}{2}-1\right) = -\frac{1}{5}$

$\therefore \boxed{h = \frac{1}{4}}$

Exercise 2B Resolution of a vector into rectangular components

→ Magnitude of 3D vector

- If $\underset{\sim}{u} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k}$, then $|\underset{\sim}{u}| = \sqrt{x^2+y^2+z^2}$

→ Unit vector ($\hat{\underset{\sim}{u}}$ is a unit vector in direction of $\underset{\sim}{u}$)

- $\hat{\underset{\sim}{u}} = \frac{1}{|\underset{\sim}{u}|}\underset{\sim}{u} = \frac{x\underset{\sim}{i}+y\underset{\sim}{j}+z\underset{\sim}{k}}{\sqrt{x^2+y^2+z^2}}$

→ Angle made by a vector with an axis

- $\underset{\sim}{a} = x\underset{\sim}{i} + y\underset{\sim}{j} + z\underset{\sim}{k} \rightarrow \alpha, \beta, \gamma$
- $\cos(\alpha) = \frac{x}{|\underset{\sim}{a}|}$, $\cos(\beta) = \frac{y}{|\underset{\sim}{a}|}$, $\cos(\gamma) = \frac{z}{|\underset{\sim}{a}|}$

→ Examples

- e.g. 1. Points A and B are defined by position vectors $\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k}$ and $\underset{\sim}{b} = 3\underset{\sim}{i} - 2\underset{\sim}{j} - \underset{\sim}{k}$ respectively. The point M is on the line segment such that AM : MB = 4 : 1.

(a) Find (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AM}$ (iii) $\overrightarrow{OM}$

(b) Find the coordinates of M.

(a) (i) $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\underset{\sim}{i} - \underset{\sim}{j} + 5\underset{\sim}{k} + 3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$

$\therefore = \boxed{2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k}}$

(ii) $\overrightarrow{AM} = \frac{4}{5}\overrightarrow{AB} = \boxed{\frac{4}{5}(2\underset{\sim}{i} - 3\underset{\sim}{j} + 4\underset{\sim}{k})}$

(iii) $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \underset{\sim}{i} + \underset{\sim}{j} - 5\underset{\sim}{k} + \frac{8}{5}\underset{\sim}{i} - \frac{12}{5}\underset{\sim}{j} + \frac{16}{5}\underset{\sim}{k}$

$= \boxed{\frac{13}{5}\underset{\sim}{i} - \frac{7}{5}\underset{\sim}{j} - \frac{9}{5}\underset{\sim}{k}}$

(b) $M = \boxed{\left(\frac{13}{5}, -\frac{7}{5}, -\frac{9}{5}\right)}$

CAS skills:

- Menu → Matrix and Vector (7)
  → Norms (7) → Norm (1)
  → norm([x y z])

- Menu → Matrix and Vector (7)
  → Vector (C) → Unit Vector (1)
  → unitV([x, y z])

e.g. 2. Let i be a unit vector in the east direction and let j be a unit vector in the north direction, with units in kilometres.

(a) Show that the unit vector in the direction N60°W is $-\frac{\sqrt{3}}{2}i + \frac{1}{2}j$.

(b) If a car drives 3km in direction N60°W, find the position vector of the car with respect to its starting point.

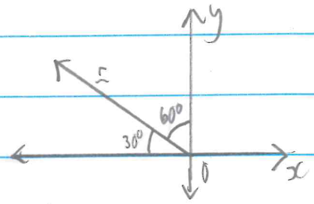
(c) The car then drives 6.5 km due north. Find:

(i) the position vector of the car

(ii) the distance of the car from starting point

(iii) the bearing of the car from starting point

(a) Let r denote unit vector in direction N60°W.



$r = -\cos(30°)\,i + \cos(60°)\,j$

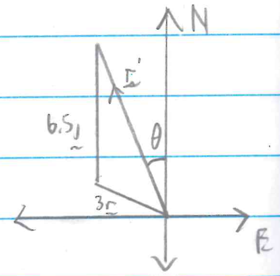
$\therefore = -\frac{\sqrt{3}}{2}i + \frac{1}{2}j$ (QED)

(b) $3r = 3\left(-\frac{\sqrt{3}}{2}i + \frac{1}{2}j\right) = \boxed{-\frac{3\sqrt{3}}{2}i + \frac{3}{2}j}$

(c)(i) Let r' denote the new position vector.

$r' = 3r + 6.5j = -\frac{3\sqrt{3}}{2}i + \frac{3}{2}j + \frac{13}{2}j$

$\therefore = \boxed{-\frac{3\sqrt{3}}{2}i + 8j}$



(ii) $|r'| = \sqrt{\left(-\frac{3\sqrt{3}}{2}\right)^2 + 8^2} = \sqrt{\frac{9 \times 3}{4} + 64}$

$\therefore = \boxed{\frac{1}{2}\sqrt{283}}$

(iii) $r' = -\frac{3\sqrt{3}}{2}i + 8j \rightarrow \tan(\theta) = \frac{3\sqrt{3}}{16}$

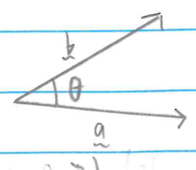
$\rightarrow \theta = \tan^{-1}\left(\frac{3\sqrt{3}}{16}\right) = 18°$

Bearing: $\boxed{342°}$ (nearest degree)

Exercise 2C Scalar (dot) product of vectors

→ Definition → product of two vectors, produce a scalar result (not vector!)

- notation: $a \cdot b$ (remember the DOT!)
- $a \cdot b = |a| \times |b| \times \cos\theta$
- N.B. θ (angle between a and b) must be $0 \le \theta \le \pi$ (180°)
  - known as tail-to-tail angle



CAS skills: Menu → Matrix and Vector (7) → Vector (C)

→ Dot Product (3) → dotP([x_1 y_1 z_1], [x_2 y_2 z_2])

→ Properties of dot product

- $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$
- $\underset{\sim}{a} \cdot 0 = 0$
- If $\underset{\sim}{a} \parallel \underset{\sim}{b}$, then $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|$ (since $\cos\theta = \cos(0) = 1$)
- $k(\underset{\sim}{a} \cdot \underset{\sim}{b}) = (k\underset{\sim}{a}) \cdot \underset{\sim}{b} = \underset{\sim}{a}(k\underset{\sim}{b})$
- If $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0$, either $\underset{\sim}{a} = 0$ or $\underset{\sim}{b} = 0$ or $\underset{\sim}{a} \perp \underset{\sim}{b}$ ($\cos(90°) = 0$)
- $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2$ (since $\cos\theta = \cos(0) = 1$)
- $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$
- $\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{cases} |\underset{\sim}{a}||\underset{\sim}{b}| & \text{if } \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are parallel and in same direction} \\ -|\underset{\sim}{a}||\underset{\sim}{b}| & \text{if } \underset{\sim}{a} \text{ and } \underset{\sim}{b} \text{ are parallel and in opposite directions} \end{cases}$

→ Dot product in $\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ form

- Let $\underset{\sim}{a} = a_x\underset{\sim}{i} + a_y\underset{\sim}{j} + a_z\underset{\sim}{k}$ and $\underset{\sim}{b} = b_x\underset{\sim}{i} + b_y\underset{\sim}{j} + b_z\underset{\sim}{k}$

  → $\underset{\sim}{a} \cdot \underset{\sim}{b} = a_x b_x + a_y b_y + a_z b_z$

→ Finding angle between 2 vectors

- $\cos(\theta) = \dfrac{a_x b_x + a_y b_y + a_z b_z}{|\underset{\sim}{a}| \times |\underset{\sim}{b}|} = \dfrac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \times |\underset{\sim}{b}|}$
- $\cos(\angle ABC) = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \times |\overrightarrow{BC}|}$

- e.g. 1. A, B and C are points defined by position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively, where $\underset{\sim}{a} = \underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{c} = \underset{\sim}{i} - 2\underset{\sim}{j} - 2\underset{\sim}{k}$. Find angle $\angle ABC$, correct to one decimal place

  → $\overrightarrow{BA} = \underset{\sim}{a} - \underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$

  → $\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b} = -\underset{\sim}{i} - 3\underset{\sim}{j} - 2\underset{\sim}{k}$

  $|\overrightarrow{BA}| = \sqrt{(-1)^2 + 2^2 + 1} = \sqrt{6}$

  $|\overrightarrow{BC}| = \sqrt{(-1)^2 + (-3)^2 + (-2)^2} = \sqrt{14}$

  $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-1)(-1) + 2(-3) + (-1)(-2) = -3$

  $\cos(\angle ABC) = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}| \times |\overrightarrow{BC}|} = \dfrac{-3}{\sqrt{6} \times \sqrt{14}} = \dfrac{-3}{2\sqrt{21}} = -\dfrac{\sqrt{21}}{14}$

  $\angle ABC = \cos^{-1}\left(-\dfrac{\sqrt{21}}{14}\right) \approx \boxed{109.1°}$ (1 dp)

- e.g. 2. Let C and D be points with position vectors $\underset{\sim}{c}$ and $\underset{\sim}{d}$ respectively. If $|\underset{\sim}{c}| = 5$, $|\underset{\sim}{d}| = 7$ and $\underset{\sim}{c} \cdot \underset{\sim}{d} = 4$, find $|\overrightarrow{CD}|$.

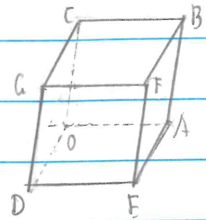
  → $\cos(\theta) = \dfrac{\underset{\sim}{c} \cdot \underset{\sim}{d}}{|\underset{\sim}{c}| \times |\underset{\sim}{d}|} = \dfrac{4}{5 \times 7} = \dfrac{4}{35}$

  $|\overrightarrow{CD}|^2 = |\underset{\sim}{c}|^2 + |\underset{\sim}{d}|^2 - 2|\underset{\sim}{c}||\underset{\sim}{d}|\cos(\theta)$ (cosine rule)

  $= 5^2 + 7^2 - 2(5)(7)\left(\dfrac{4}{35}\right) = 66$

  $|\overrightarrow{CD}| = \boxed{\sqrt{66}}$

e.g.3. A parallelepiped is an oblique prism that has a parallelogram cross-section. It has three pairs of parallel and congruent faces. OABCDEFG is a parallelepiped with $OA = 3\underset{\sim}{j}$, $\overrightarrow{OC} = -\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\overrightarrow{OD} = 2\underset{\sim}{i} - \underset{\sim}{j}$. Show that the diagonals DB and CE bisect each other, and find the acute angle between them.



$\overrightarrow{CE} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{AE} = -\overrightarrow{OC} + \overrightarrow{OA} + \overrightarrow{OD}$

$= -(\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k}) + 3\underset{\sim}{j} + 2\underset{\sim}{i} - \underset{\sim}{j}$

$= 3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$

$\overrightarrow{DB} = \overrightarrow{DO} + \overrightarrow{OA} + \overrightarrow{AB} = -\overrightarrow{OD} + \overrightarrow{OA} + \overrightarrow{OC}$

$= -(2\underset{\sim}{i} - \underset{\sim}{j}) + 3\underset{\sim}{j} + (-\underset{\sim}{i} + \underset{\sim}{j} + 2\underset{\sim}{k})$

$= -3\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k}$

Let M be midpoint of CE.

$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CE} = \frac{1}{2}(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$

$\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB} = -\frac{1}{2}(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) + 3\underset{\sim}{j}$

$= -\frac{3}{2}\underset{\sim}{i} + \frac{5}{2}\underset{\sim}{j} + \underset{\sim}{k}$

$\overrightarrow{DM} = \overrightarrow{DE} + \overrightarrow{EM} = \overrightarrow{OA} - \overrightarrow{CM}$

$= 3\underset{\sim}{j} - \frac{1}{2}(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k})$

$= -\frac{3}{2}\underset{\sim}{i} + \frac{5}{2}\underset{\sim}{j} + \underset{\sim}{k} = \overrightarrow{MB}$

$\rightarrow$ M is midpoint of DB

$\therefore$ $\boxed{\text{Diagonals DB and CE bisect each other}}$ (as required)

$\cos(\theta) = \frac{\overrightarrow{CE} \cdot \overrightarrow{DB}}{|\overrightarrow{CE}||\overrightarrow{DB}|} = \frac{(3\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}) \cdot (-3\underset{\sim}{i} + 5\underset{\sim}{j} + 2\underset{\sim}{k})}{\sqrt{14} \times \sqrt{38}}$

$= \frac{-9 + 5 - 4}{2\sqrt{133}} = \frac{-4}{\sqrt{133}}$

$\rightarrow \theta = \cos^{-1}\left(\frac{-4\sqrt{133}}{133}\right) = 110.29°$

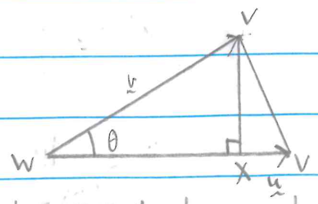
Acute angle $= 180° - 110.29° = \boxed{69.71°}$

Exercise 2D Vector projections

$\rightarrow$ Scalar resolute, $\underset{\sim}{v} \parallel$ to $\underset{\sim}{u} = \underset{\sim}{v} \cdot \hat{\underset{\sim}{u}}$

$\rightarrow$ Vector resolute, $\underset{\sim}{v} \parallel$ to $\underset{\sim}{u} = (\underset{\sim}{v} \cdot \hat{\underset{\sim}{u}})\hat{\underset{\sim}{u}} = \left(\frac{\underset{\sim}{v} \cdot \underset{\sim}{u}}{\underset{\sim}{u} \cdot \underset{\sim}{u}}\right)\underset{\sim}{u}$

$\rightarrow$ Vector resolute, $\underset{\sim}{v} \perp$ to $\underset{\sim}{u} = \underset{\sim}{v} - \underset{\sim}{v}_{\parallel \underset{\sim}{u}}$



N.B. $\underset{\sim}{u}$ = parallel resolute along stated direction + perpendicular resolute along stated direction

→ Application Questions

- e.g. 1. $\underset{\sim}{a} = 2\underset{\sim}{i} - 3\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k}$, $\underset{\sim}{b} = -\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}$

Find (a) the scalar resolute of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

(b) the vector resolute of $\underset{\sim}{a}$ onto $\underset{\sim}{b}$

(c) the vector resolute of $\underset{\sim}{a}$ perpendicular to $\underset{\sim}{b}$

(a) $\underset{\sim}{a} \cdot \underset{\sim}{b} = 2(-1) - 3(2) - \sqrt{3}(-3) = 3\sqrt{3} - 8$

$\rightarrow \underset{\sim}{a} \cdot \hat{\underset{\sim}{b}} = \frac{3\sqrt{3}-8}{\sqrt{14}} = \boxed{\frac{\sqrt{14}(3\sqrt{3}-8)}{14}}$

(b) $\left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} = \left(\frac{3\sqrt{3}-8}{1+4+9}\right)(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k}) = \boxed{\left(\frac{3\sqrt{3}-8}{14}\right)(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})}$

(c) $\underset{\sim}{a}_{\perp \underset{\sim}{b}} = \underset{\sim}{a} - \underset{\sim}{a}_{\parallel \underset{\sim}{b}} = \boxed{2\underset{\sim}{i} - 3\underset{\sim}{j} - \sqrt{3}\underset{\sim}{k} - \left(\frac{3\sqrt{3}-8}{14}\right)(-\underset{\sim}{i} + 2\underset{\sim}{j} - 3\underset{\sim}{k})}$

- e.g. 2. Let $\underset{\sim}{a} = -\frac{7\sqrt{3}}{3}\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{b} = \underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k}$ (VCAA 2013 Exam 2 Q4 a-d)

(a) Find a unit vector in direction of $\underset{\sim}{b}$. (1 mark)

(b) Resolve $\underset{\sim}{a}$ into two vector components, one that is parallel to $\underset{\sim}{b}$ and one that is perpendicular to $\underset{\sim}{b}$. (3 marks)

(c) Find the value of m such that $\underset{\sim}{c} = m\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}$ makes an angle of $\frac{2\pi}{3}$ with $\underset{\sim}{b}$ and where $\underset{\sim}{c} \neq \underset{\sim}{a}$. (2 marks)

(d) Find the angle, in degrees, that $\underset{\sim}{c}$ makes with $\underset{\sim}{a}$. (2 marks)

(a) $|\underset{\sim}{b}| = \sqrt{1+3+12} = 4$

$\hat{\underset{\sim}{b}} = \frac{1}{4}(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k})$

(b) $\underset{\sim}{a}_{\parallel \underset{\sim}{b}} = \left(\frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{\underset{\sim}{b} \cdot \underset{\sim}{b}}\right)\underset{\sim}{b} = \left(\frac{-\frac{7\sqrt{3}}{3} + \sqrt{3} - 4\sqrt{3}}{1+3+12}\right)(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k})$

$= \left(-\frac{16\sqrt{3}}{3} \times \frac{1}{16}\right)(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k})$

$= -\frac{\sqrt{3}}{3}(\underset{\sim}{i} + \sqrt{3}\underset{\sim}{j} + 2\sqrt{3}\underset{\sim}{k})$

$\boxed{\underset{\sim}{a}_{\parallel \underset{\sim}{b}} = -\frac{\sqrt{3}}{3}\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k}}$

$\underset{\sim}{a}_{\perp \underset{\sim}{b}} = -\frac{7\sqrt{3}}{3}\underset{\sim}{i} + \underset{\sim}{j} - 2\underset{\sim}{k} - \left(-\frac{\sqrt{3}}{3}\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}\right)$

$\boxed{\underset{\sim}{a}_{\perp \underset{\sim}{b}} = -2\sqrt{3}\underset{\sim}{i} + 2\underset{\sim}{j}}$

(c) $|\underset{\sim}{b}| = 4$, $|\underset{\sim}{c}| = \sqrt{m^2+1+4} = \sqrt{m^2+5}$

$\underset{\sim}{b} \cdot \underset{\sim}{c} = m + \sqrt{3} + 2\sqrt{3}(-2) = m - 3\sqrt{3}$

$\rightarrow \underset{\sim}{b} \cdot \underset{\sim}{c} = |\underset{\sim}{b}||\underset{\sim}{c}|\cos\left(\frac{2\pi}{3}\right) \rightarrow m - 3\sqrt{3} = 4\sqrt{m^2+5} \times \cos\left(\frac{2\pi}{3}\right)$

Solve for m, $m = -\frac{7\sqrt{3}}{3}$ (reject) or $m = \frac{\sqrt{3}}{3}$ (accept) (since $\underset{\sim}{c} \neq \underset{\sim}{a}$) $\rightarrow \boxed{m = \frac{\sqrt{3}}{3}}$

(d) $|\underset{\sim}{a}| = \sqrt{\frac{49}{3}+1+4} = \frac{8\sqrt{3}}{3}$ $|\underset{\sim}{c}| = \sqrt{\frac{1}{3}+1+4} = \frac{4\sqrt{3}}{3}$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = \left(-\frac{7\sqrt{3}}{3}\right)\left(\frac{\sqrt{3}}{3}\right) + 1(1) + 2(-2) = -\frac{7}{3} + 1 + 4 = \frac{8}{3}$

$\underset{\sim}{a} \cdot \underset{\sim}{c} = |\underset{\sim}{a}| \times |\underset{\sim}{c}| \times \cos\theta \rightarrow \frac{8}{3} = \frac{8\sqrt{3}}{3} \times \frac{4\sqrt{3}}{3} \times \cos(\theta)$

$\cos(\theta) = \frac{1}{4} \rightarrow \boxed{\theta = 75.5°}$ (1 decimal place) e.g. 3. Points A, B and C have position vectors $\underset{\sim}{a} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{c} = 2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}$. Find:

(a) (i) $\overrightarrow{AB}$ (ii) $\overrightarrow{AC}$

(b) the vector resolute of $\overrightarrow{AB}$ in direction of $\overrightarrow{AC}$

(c) the shortest distance from B to line AC

(d) the area of triangle ABC

(a) (i) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a} = (2\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

$= \boxed{\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}}$

(ii) $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a} = (2\underset{\sim}{i} - 3\underset{\sim}{j} + \underset{\sim}{k}) - (\underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k})$

$= \boxed{\underset{\sim}{i} - 5\underset{\sim}{j}}$

(b) $\left(\dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC} \cdot \overrightarrow{AC}}\right)\overrightarrow{AC} = \dfrac{1 + (-1)(-5)}{1^2 + (-5)^2}(\underset{\sim}{i} - 5\underset{\sim}{j})$

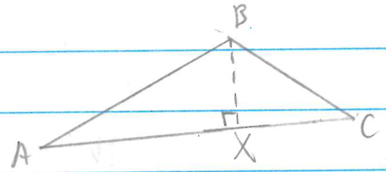
$= \boxed{\frac{3}{13}(\underset{\sim}{i} - 5\underset{\sim}{j})}$

(c) Perpendicular component: $\overrightarrow{AB} - \frac{3}{13}(\underset{\sim}{i} - 5\underset{\sim}{j}) = (\underset{\sim}{i} - \underset{\sim}{j} - 2\underset{\sim}{k}) - (\frac{3}{13}\underset{\sim}{i} - \frac{15}{13}\underset{\sim}{j})$

$= \frac{10}{13}\underset{\sim}{i} + \frac{2}{13}\underset{\sim}{j} - 2\underset{\sim}{k}$

$\rightarrow$ Magnitude $= \sqrt{\left(\frac{10}{13}\right)^2 + \left(\frac{2}{13}\right)^2 + (-2)^2}$

$= \boxed{\frac{2\sqrt{195}}{13} \text{ units}}$

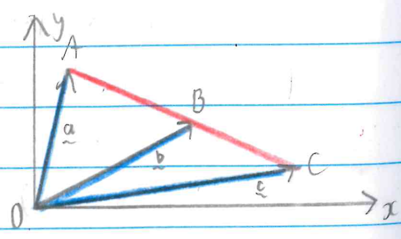


(d) $|\overrightarrow{AC}| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$

Area $\Delta ABC = \frac{1}{2} \times \sqrt{26} \times \frac{2\sqrt{195}}{13}$

$= \boxed{\sqrt{30} \text{ units}^2}$

Exercise 2E Collinearity



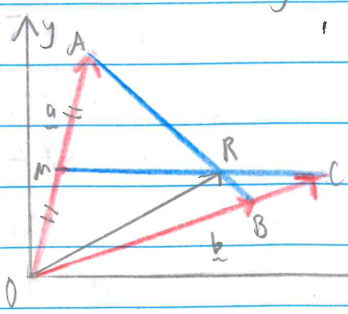
$\rightarrow$ Collinearity (when three or more points lie on same straight line)

- $\overrightarrow{AB} = k\overrightarrow{AC}$ · $\overrightarrow{AB} \parallel \overrightarrow{BC}$
- $\underset{\sim}{b} = (1-k)\underset{\sim}{a} + k\underset{\sim}{c}$

e.g. 1. Let $\overrightarrow{OA} = \underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$ where vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are linearly independent. Let M be midpoint of OA, C be point such that $\overrightarrow{OC} = \frac{4}{3}\overrightarrow{OB}$ and R be point of intersection of lines AB and MC

(a) Find $\overrightarrow{OR}$ in terms of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

(b) Hence find AR : RB



OB : BC = 3 : 1

(a) Let $\overrightarrow{MR} = k\overrightarrow{MC}$ and $\overrightarrow{AR} = l\overrightarrow{AB}$.

$\overrightarrow{AB} = -\underset{\sim}{a} + \underset{\sim}{b}$, $\overrightarrow{MC} = -\frac{1}{2}\underset{\sim}{a} + \frac{4}{3}\underset{\sim}{b}$

$$\overrightarrow{OR} = \overrightarrow{OM} + \overrightarrow{MR} = \frac{1}{2}\underset{\sim}{a} + k\left(-\frac{1}{2}\underset{\sim}{a} + \frac{4}{3}\underset{\sim}{b}\right) = \frac{1}{2}\underset{\sim}{a}(1-k) + \frac{4k}{3}\underset{\sim}{b}$$

$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR} = \underset{\sim}{a} + l(-\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{a}(1-l) + l\underset{\sim}{b}$$

Equating like terms: $\frac{1}{2}(1-k) = 1 - l \cdots ①$

$\frac{4k}{3} = l \cdots ②$

Sub ② into ①, $\frac{1}{2} - \frac{k}{2} = 1 - \frac{4k}{3}$

$3 - 3k = 6 - 8k$

$5k = 3 \rightarrow k = \frac{3}{5}$

$\rightarrow l = \frac{4}{3}k = \frac{4}{3} \times \frac{3}{5} = \frac{4}{5}$

$\overrightarrow{OR} = \underset{\sim}{a}(1-l) + l\underset{\sim}{b}$

$= \underset{\sim}{a}\left(1 - \frac{4}{5}\right) + \frac{4}{5}\underset{\sim}{b} = \boxed{\frac{1}{5}\underset{\sim}{a} + \frac{4}{5}\underset{\sim}{b}}$

(b) $AR : RB = l : 1-l$ (from $\overrightarrow{AR} = l\overrightarrow{AB}$)

$= \frac{4}{5} : \frac{1}{5} = \boxed{4:1}$

e.g. 2. In triangle OAB, $\overrightarrow{OA} = 3\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}$

(a) Use the scalar product to show that $\angle AOB$ is an obtuse angle.

(b) Find $\overrightarrow{OP}$, where P is:

(i) the midpoint of $\overrightarrow{AB}$

(ii) the point on AB such that OP is perpendicular to AB.

(iii) the point where bisector of $\angle AOB$ intersects AB.

(a) $\overrightarrow{OA} \cdot \overrightarrow{OB} = |\overrightarrow{OA}| \times |\overrightarrow{OB}| \cos(\theta)$

$3(1) + 4(-2) = \left(\sqrt{3^2+4^2}\right) \times \left(\sqrt{1^2+2^2+(-2)^2}\right) \times \cos(\theta)$

$-5 = 5 \times 3\cos(\theta) \rightarrow \cos(\theta) = -\frac{1}{3}$

$\therefore$ $\boxed{\theta \text{ is obtuse}}$ (as required)

(b) (i) $\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB}$

$= \overrightarrow{OA} + \frac{1}{2}(\overrightarrow{OB} - \overrightarrow{OA})$

$= \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB})$

$= \frac{1}{2}(3\underset{\sim}{i} + 4\underset{\sim}{k} + \underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) = \boxed{2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}}$

(ii) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= (\underset{\sim}{i} + 2\underset{\sim}{j} - 2\underset{\sim}{k}) - (3\underset{\sim}{i} + 4\underset{\sim}{k})$

$= -2\underset{\sim}{i} + 2\underset{\sim}{j} - 6\underset{\sim}{k}$