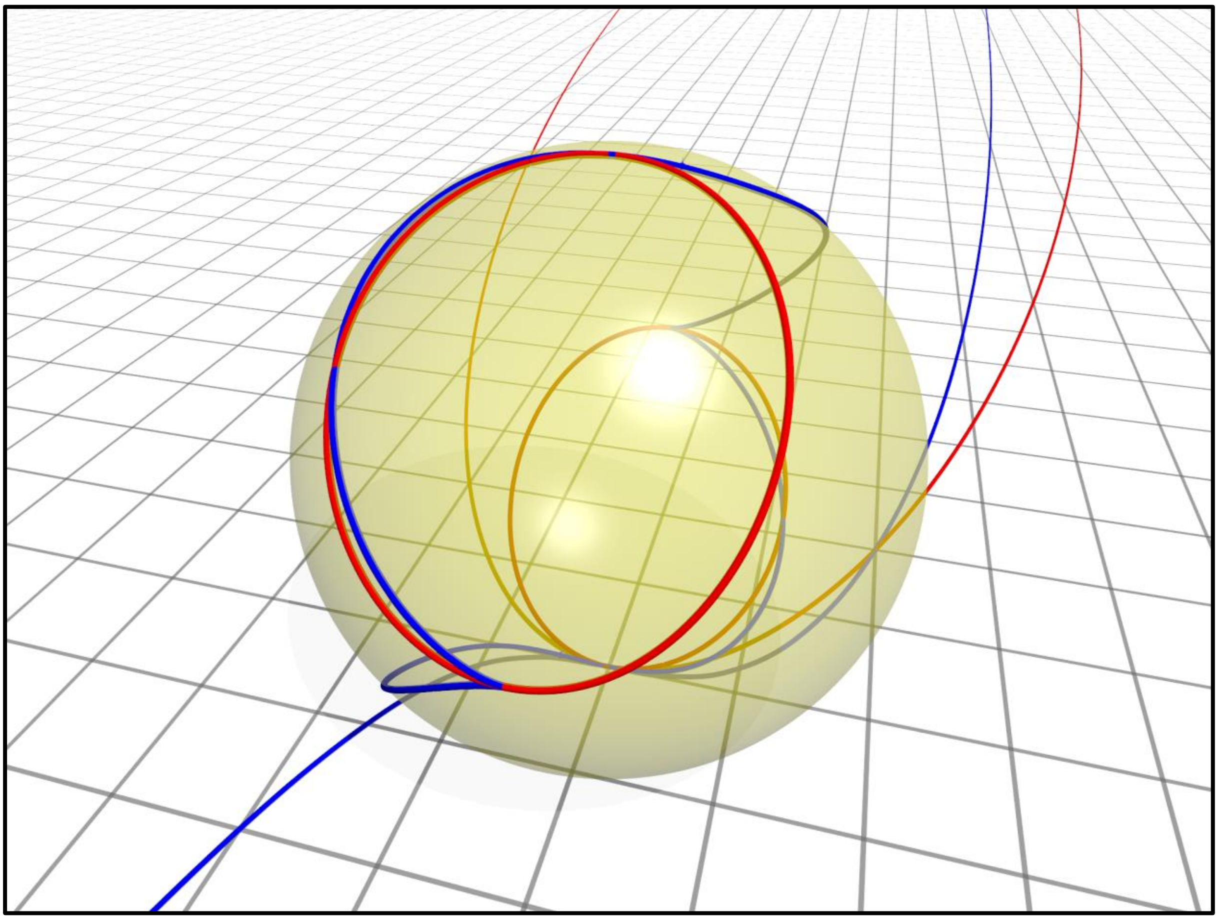
UNIT 3 SPECIALIST MATHS

DYNAMICS



WRITTEN BY A STUDENT WHO OBTAINED A SCALED STUDY SCORE OF 52.46!

Chapter 13 Dynamics

Exercise 13A Force

→ Force → push or pull; vector (magnitude + direction)

- resolved into rectangular components in x, y, z
- Units → Newtons (N) → force req. to accelerate 1kg m by 1 m/s^2
  - → kilogram weight (kg wt) → 1 kg wt = g N, 1 N = $\frac{1}{g}$ kg N
- Mks system → length (metre), mass (kg), time (second)

→ Resultant force (vector sum of forces acting at a point)

- N.B. use addition of vectors (see Ch 2)

  sin rule: $\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)}$ one side + two angles / two sides + non-included angle

  cos rule: $c^2 = a^2 + b^2 - 2ab\cos(C)$ two sides + included angle / three sides

- Initial rule: let $\underset{\sim}{i}$ be unit vector in east direction respect to (force)

  Let $\underset{\wedge}{j}$ be unit vector in north direction of (force)

  let $\underset{\sim}{k}$ be unit vector in upwards direction of (force)

Exercise 13B Newton's laws of motion

→ Weight: W = mg N = m kg wt ($g = 9.8$ m/s^2)

→ Momentum: $\underset{\sim}{p} = m\underset{\sim}{v}$ kg m/s

→ Newton's three laws of motion

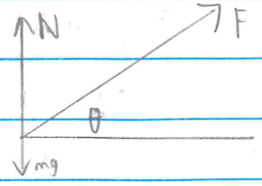
- 1st: particle remains stationary, or in uniform straight line motion (constant velocity) unless acted on by some overall external force
  - forces must be balanced if body at rest or uniform straight line motion ($a = 0$)
- 2nd: rate of change of momentum with respect to time proportional to resultant force ($a \neq 0$)

  $\underset{\sim}{F}_{net} = m\underset{\sim}{a}$ or $\sum \underset{\sim}{F} = m\underset{\sim}{a}$ (N → kg m/s^2)
- 3rd: if A exerts force on B → B exerts collinear force of equal magnitude + opposite direction on A

  $\underset{\sim}{F}_{A \text{ on } B} = -\underset{\sim}{F}_{B \text{ on } A}$

→ Normal reaction force (R/N)

- $N = mg$ (no vertical motion)
- $N - mg = ma$ (forces acting on particle)
- $N = mg - F\sin(\theta)$ (acts at angle)



→ Sliding friction: $F_R = \mu R$ ($\mu \to$ coefficient of friction)

- opposes motion; dependent upon roughness of surface (μ)
  normal force (N)

→ Applying Newton's Law of Motion

① Draw a clear diagram showing all objects comprising system
② Identify all forces acting on each object (with clear labeled arrows)
③ Resolve each force into two perpendicular components (if necessary)
N.B. For objects in contact with a surface, choose two components to be in directions parallel + perpendicular to surface
$(\underset{\sim}{F} = |\underset{\sim}{F}|\cos(\theta)\underset{\sim}{i} + |\underset{\sim}{F}|\sin(\theta)\underset{\sim}{j})$ (see Ex 13C)
④ For each direction, write down an expression for net force acting on each object in terms of individual forces acting on that object.
⑤ Apply Newton's 2nd law to get equations of motion for each object.

N.B. $\underset{\sim}{F}_{net}$ constant → 'a' constant → straight line motion formulae, differential eqn used
$\underset{\sim}{F}_{net}$ variable → 'a' not constant → differential eqn involving acceleration set up + solved

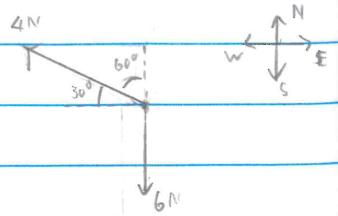
- Spring balance (tension force), Bathroom scales (normal reaction force)

- e.g. A particle is acted on by two forces, one of 6 newtons acting due south, the other of 4 newtons acting in the direction N 60° W (VCAA 2012 Exam 2 Q14)
  Find the magnitude of the resultant force, in newtons, acting on particle.

→ North - South: $4\cos(60°) - 6 = -4$ N → 4N south

East - West: $4\cos(30°) = 2\sqrt{3}$ N west

$|F_{net}| = \sqrt{4^2 + (2\sqrt{3})^2} = \sqrt{16+12} = \boxed{2\sqrt{7} \text{ N}}$



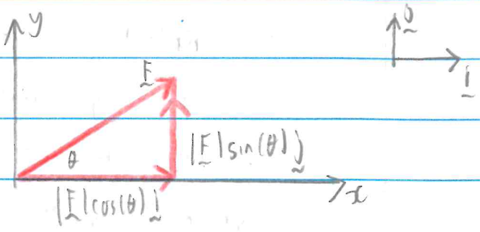
Exercise 13C Resolution of forces and inclined planes

→ Resolution of forces

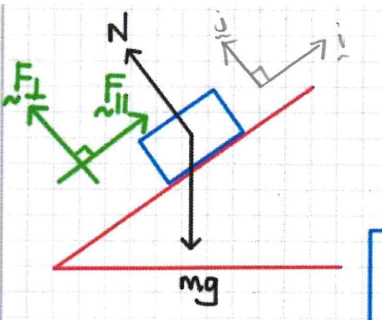
- $\underset{\sim}{F} = |\underset{\sim}{F}|\cos(\theta)\underset{\sim}{i} + |\underset{\sim}{F}|\sin(\theta)\underset{\sim}{j}$

  $\underset{\sim}{i}$ - comp. parallel to x-axis

  $\underset{\sim}{j}$ - comp. parallel to y-axis



→ Normal reaction forces for inclined planes



* The Normal Reaction Force is always at right angles to the plane.

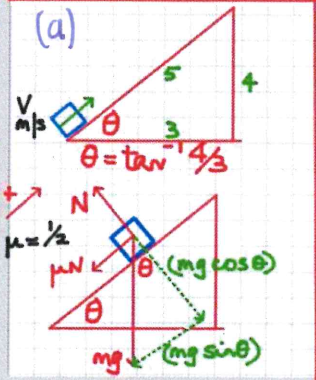
* We are concerned with motion along (parallel to) the plane.

Hence, it becomes convenient to resolve the forces parallel and perpendicular to the plane in inclined plane problems, rather than horizontally and vertically.

Example 16

A slope is inclined at an angle θ to the horizontal where $\tan\theta = \frac{4}{3}$. A particle is projected from the foot of the slope up a line of greatest slope with a speed of V m/s and comes instantaneously to rest after travelling 6m. If the coefficient of friction between the particle and the slope is $\frac{1}{2}$, calculate:

a the value of V **b** the speed of the particle when it returns to its starting point



So $\sin\theta = 4/5$, $\cos\theta = 3/5$ from the triangle opposite.

We need to calculate the net force and hence acceleration up the plane:—

$\sum \underset{\sim}{F}_\perp = 0 \rightarrow mg\cos\theta = N$

ie $N = \frac{3mg}{5}$

$\sum \underset{\sim}{F}_{\parallel} = ma$

$\rightarrow -mg\sin\theta - \mu N = ma$

$\rightarrow -\frac{4\cancel{m}g}{5} - \frac{1}{2}\times\frac{3\cancel{m}g}{5} = \cancel{m}a$

$-\frac{4g}{5} - \frac{3g}{10} = a$

$\rightarrow a = -\frac{11g}{10}$ m/sec^2

Now for constant acceleration:—

$v^2 = u^2 + 2ax$ where

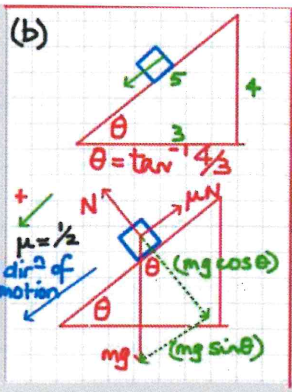
$u = V$, $v = 0$, $a = -\frac{11g}{10}$, $x = 6$

$\rightarrow 0 = V^2 + 2\times\frac{-11g}{10}\times 6$

$V^2 = \frac{66g}{5} \rightarrow V = +\sqrt{66g/5}$

(has to be +ve, moving UP the plane)

$\boxed{\cong 11{\cdot}37 \text{ m/sec (2 dp)}}$



When the particle moves down the plane, it has a different net force acting on it, and hence, a different acceleration.

N is the same as when moving up the plane.

Let DOWN the plane be +ve dir^n.

So $\sum \underset{\sim}{F}_{\parallel} = ma$

$\rightarrow mg\sin\theta - \mu N = ma$

$\rightarrow \cancel{m}g\times\frac{4}{5} - \frac{1}{2}\times\frac{3}{5}\cancel{m}g = \cancel{m}a$

$\rightarrow g\times\frac{4}{5} - \frac{3g}{10} = a$

$\rightarrow \frac{g}{2} = a$

Now for constant acceleration down the plane :—

$v^2 = u^2 + 2ax$ where

$v = ?$, $u = 0$, $a = \frac{g}{2}$, $x = 6$

So, $v^2 = 0 + \cancel{2}\times\frac{g}{\cancel{2}}\times 6$

$= 6g$

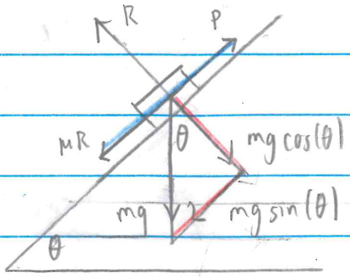
$\rightarrow v = +\sqrt{6g}$

(must be +ve as particle moves DOWN the plane).

$\cong \boxed{7{\cdot}67 \text{ m/s (2 dp)}}$

→ Complex example

A particle of mass m kg is being accelerated up a rough inclined plane, with coefficient of friction μ, at a m/s^2 by a force of P N acting parallel to the plane. The plane is inclined at an angle of θ to the horizontal. Find a in terms of P, θ, m, μ and g.



→ Vertical: $\sum \underset{\sim}{F}_{\perp} = 0 \to R = mg\cos(\theta)$

→ Horizontal: $\sum \underline{F}_{\parallel} = m\underset{\sim}{a} \to P - \mu R - mg\sin(\theta) = ma$

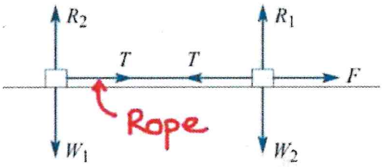
$$a = \frac{1}{m}\left(P - \mu mg\cos(\theta) - mg\sin(\theta)\right)$$

$$a = \boxed{\frac{P}{m} - \mu g\cos(\theta) - g\sin(\theta)}$$

Exercise 13D Connected particles

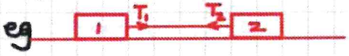
The following are examples of connected particles. Diagrams are given and the forces shown.

Two particles connected by a taut rope moving on a smooth plane.

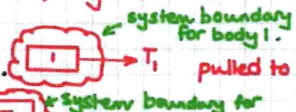


NB: The taut, inelastic rope or string means that the tension is constant throughout the length of the rope or string.

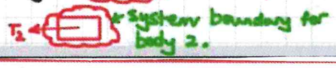
NB: The direction of the tension force, T, is always taken as away from the body they affect.

eg 

$T_1 = T_2$ T_1 pulls body 1 to the right. T_2 pulls body 2 to the left.] but it is the same, single force!

To understand this, imagine you were body 1. 

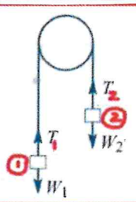
pulled to the right.

– imagine you were body 2. pulled to the left. 

A smooth light pulley (i.e. the weight of the pulley is considered negligible and the friction between rope and pulley is negligible).

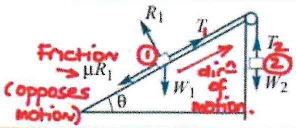
The tension in both sections of the rope can be assumed to be equal.

$T_1 = T_2$

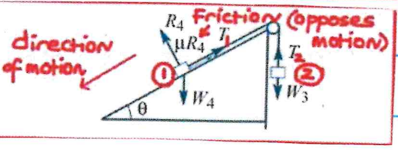


The tension in the string is of equal magnitude in both sections.

The inclined plane is rough. The body on the inclined plane is accelerating *up* the plane. $T_1 = T_2$

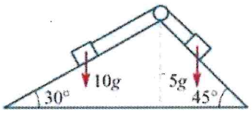


The body is accelerating *down* the inclined plane.



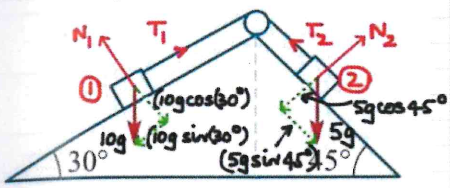
→ Complex example

7 Two masses of 10 kg and 5 kg are placed on smooth inclines of 30° and 45°, placed back to back. The masses are connected by a light string over a smooth pulley at the top of the plane.



a Find the acceleration of the system.

b Find the tension in the string.



(a) Looking at an equivalent, whole system with only external forces considered acting in direction of motion (opposing each other)

Assume $\xrightarrow{+}$ direction of motion:—

$5g\sin 45° - 10g\sin 30° = 15a$

$\rightarrow \frac{5g}{\sqrt{2}} - \frac{10g}{2} = 15a$

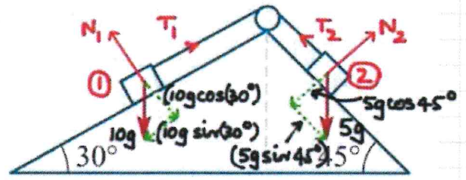
$\rightarrow \frac{5\sqrt{2}g}{2} - 5g = 15a$

$\rightarrow (5\sqrt{2} - 10)g = 30a$

$\rightarrow a = \frac{-\cancel{5}^{1}(2-\sqrt{2})g}{\cancel{30}\,6}$

$\rightarrow$ $\boxed{a = \frac{-(2-\sqrt{2})g}{6}}$ i.e. from right to left for body ①

$= 0.956\ m/sec^2$ (3dp)



(b) Taking a system boundary around ①:—

ΣF down the plane $= ma$

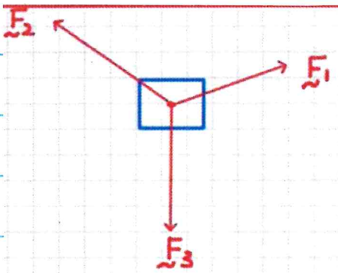
$\rightarrow 10g\sin 30° - T_1 = \frac{10 \times (2-\sqrt{2})g}{6}$

$\rightarrow \boxed{T_1 = 5g - \frac{5g}{3}(2-\sqrt{2}) = 39.43\,N \text{ (2 dp)}}$

Exercise 13F Equilibrium

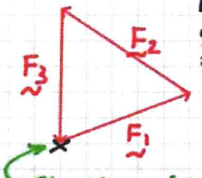
→ Equilibrium ($\underset{\sim}{F}_{net} = 0$ $\underset{\sim}{a} = 0$)

· either at rest or moves at constant velocity



If the body is in equilibrium,
$\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 0$

ie there is NO net force.



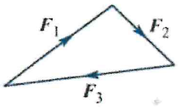
If the body is in equilibrium, $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = 0$, ie no net vector result after adding the three vectors, nose to tail!

Starting point = finishing point.

→ Triangle of forces

Triangle of forces

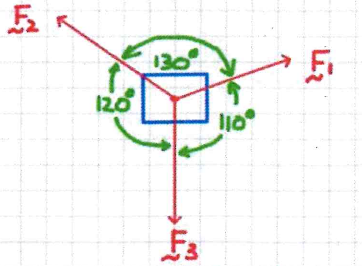
If three forces F_1, F_2 and F_3 act on a particle such that the resultant force is zero, the situation can be represented by the vector diagram shown.



This represents the vector equation $F_1 + F_2 + F_3 = 0$.

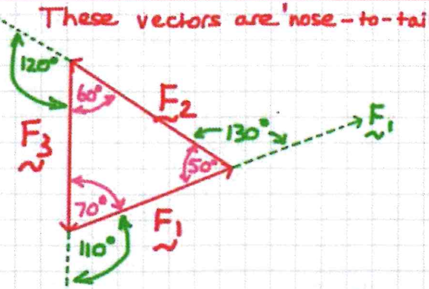
This can of course be generalised to any number of vectors by using a suitable polygon.

These vectors are 'tail-to-tail'



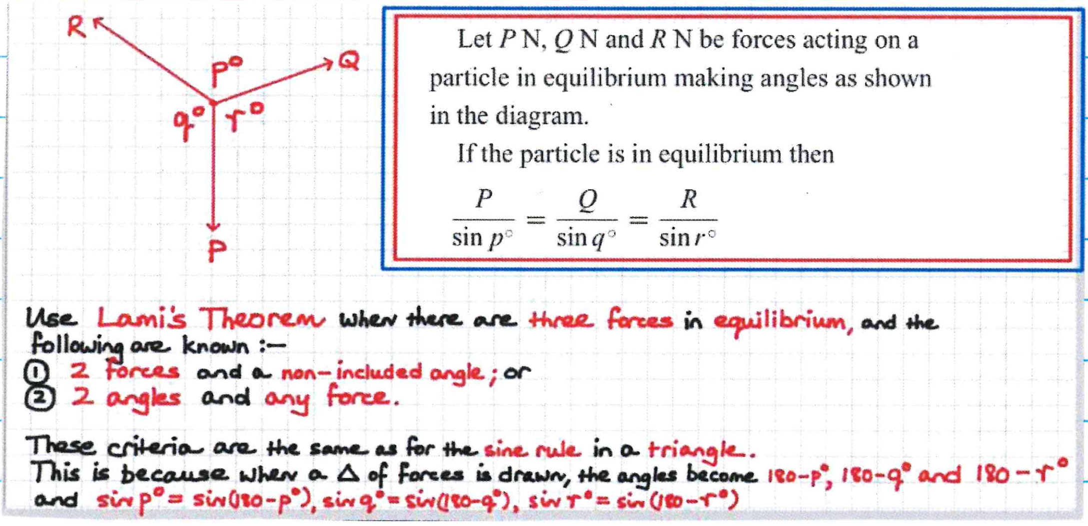
These are the actual angles between the vectors – 'tail-to-tail'

These vectors are 'nose-to-tail'.



$\underset{\sim}{F}_3$ The angles in the 'triangle of vectors' are the supplement of the angles between the vectors – 'nose-to-tail'.

→ Lami's theorem



Let P N, Q N and R N be forces acting on a particle in equilibrium making angles as shown in the diagram.

If the particle is in equilibrium then

$$\frac{P}{\sin p^\circ} = \frac{Q}{\sin q^\circ} = \frac{R}{\sin r^\circ}$$

Use Lami's Theorem when there are three forces in equilibrium, and the following are known :—

① 2 forces and a non-included angle ; or

② 2 angles and any force.

These criteria are the same as for the sine rule in a triangle.
This is because when a Δ of forces is drawn, the angles become $180-p^\circ$, $180-q^\circ$ and $180-r^\circ$ and $\sin p^\circ = \sin(180-p^\circ)$, $\sin q^\circ = \sin(180-q^\circ)$, $\sin r^\circ = \sin(180-r^\circ)$

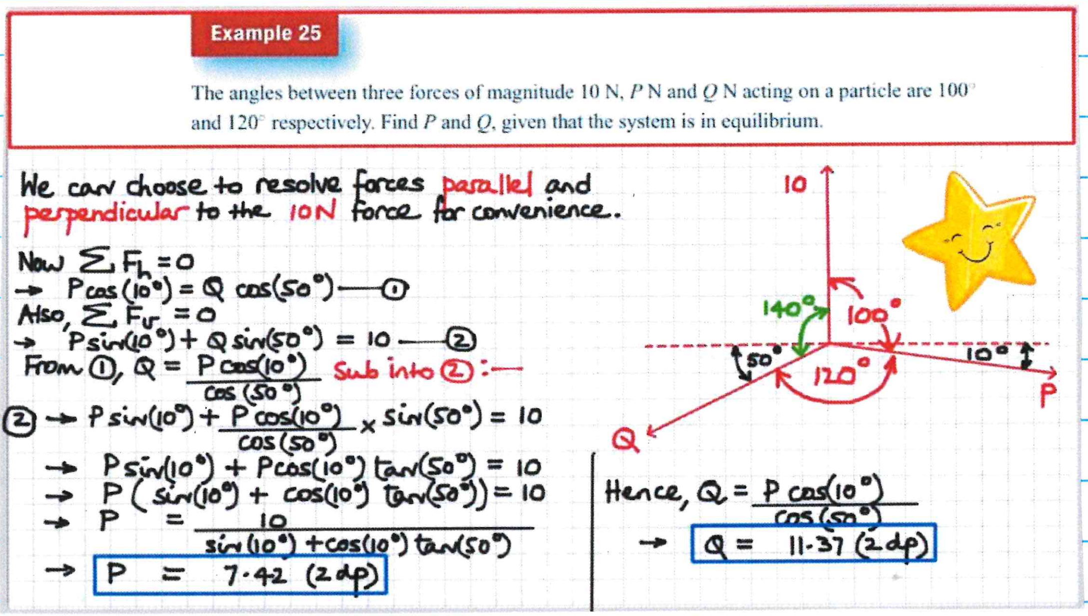
Warning ! when finding θ btw two forces → ambiguous case may arise

not always give answers expressible entirely in surd form

→ Resolution of forces → resolved into rectangular components → solve for various forces

- $\underset{\sim}{F}_1 + \underset{\sim}{F}_2 + \underset{\sim}{F}_3 = \underset{\sim}{0}$ if and only if $a_1 + a_2 + a_3 = 0$ and $b_1 + b_2 + b_3 = 0$

$\underset{\sim}{F}_1 = a_1\underset{\sim}{i} + b_1\underset{\sim}{j}$, $\underset{\sim}{F}_2 = a_2\underset{\sim}{i} + b_2\underset{\sim}{j}$, $\underset{\sim}{F}_3 = a_3\underset{\sim}{i} + b_3\underset{\sim}{j}$



Example 25

The angles between three forces of magnitude 10 N, P N and Q N acting on a particle are 100° and 120° respectively. Find P and Q, given that the system is in equilibrium.

We can choose to resolve forces parallel and perpendicular to the 10N force for convenience.

Now $\sum F_h = 0$

$\rightarrow P\cos(10^\circ) = Q\cos(50^\circ)$ —— ①

Also, $\sum F_v = 0$

$\rightarrow P\sin(10^\circ) + Q\sin(50^\circ) = 10$ —— ②

From ①, $Q = \dfrac{P\cos(10^\circ)}{\cos(50^\circ)}$ Sub into ② :—

② $\rightarrow P\sin(10^\circ) + \dfrac{P\cos(10^\circ)}{\cos(50^\circ)} \times \sin(50^\circ) = 10$

$\rightarrow P\sin(10^\circ) + P\cos(10^\circ)\tan(50^\circ) = 10$

$\rightarrow P(\sin(10^\circ) + \cos(10^\circ)\tan(50^\circ)) = 10$

$\rightarrow P = \dfrac{10}{\sin(10^\circ) + \cos(10^\circ)\tan(50^\circ)}$

$\rightarrow \boxed{P = 7{\cdot}42 \text{ (2dp)}}$

Hence, $Q = \dfrac{P\cos(10^\circ)}{\cos(50^\circ)}$

$\rightarrow \boxed{Q = 11{\cdot}37 \text{ (2dp)}}$

Exercise 13G Vector Functions (also in Ex 12D)

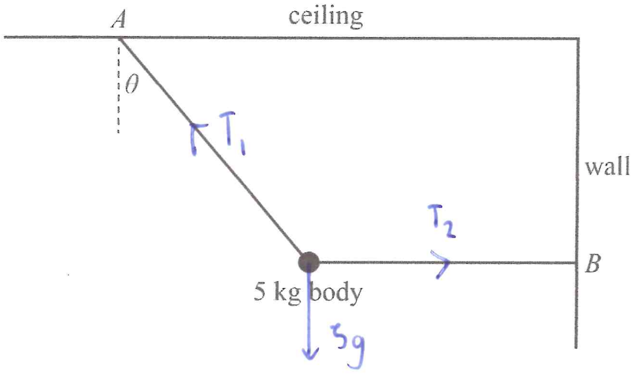
- use of $\underset{\sim}{F}_{net} = m\underset{\sim}{a}$

→ Application on Forces Questions

- e.g. 1. VCAA 2014 Exam 1 Q 8

Question 8 (7 marks)

A body of mass 5 kg is held in equilibrium by two light inextensible strings. One string is attached to a ceiling at *A* and the other to a wall at *B*. The string attached to the ceiling is at an angle θ to the vertical and has tension T_1 newtons, and the other string is horizontal and has tension T_2 newtons. Both strings are made of the same material.



a. **i.** Resolve the forces on the body vertically and horizontally, and express T_1 in terms of θ. 2 marks

Horizontal: $\Sigma F_h = 0 \rightarrow \boxed{T_2 = T_1 \sin(\theta)}$

Vertical: $\Sigma F_v = 0 \rightarrow T_1 \cos(\theta) = 5g$

$\rightarrow \boxed{T_1 = \frac{5g}{\cos(\theta)}}$

ii. Express T_2 in terms of θ. 1 mark

$T_2 = \frac{5g}{\cos(\theta)} \times \sin(\theta)$

$= \boxed{5g\tan(\theta)}$

b. Show that $\tan(\theta) < \sec(\theta)$ for $0 < \theta < \frac{\pi}{2}$. 1 mark

$\sec(\theta) - \tan(\theta) = \frac{1}{\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} = \frac{1-\sin(\theta)}{\cos(\theta)}$

If $0 < \theta < \frac{\pi}{2}$, $1-\sin(\theta) > 0$, and $\cos(\theta) > 0$ (Shown)

$\rightarrow \frac{1-\sin(\theta)}{\cos(\theta)} > 0 \rightarrow \sec(\theta) - \tan(\theta) > 0 \rightarrow \boxed{\tan(\theta) < \sec(\theta)}$ (for $0<\theta<\frac{\pi}{2}$)

c. The type of string used will break if it is subjected to a tension of more than 98 N.

Find the maximum allowable value of θ so that **neither** string will break. 3 marks

$T_1 = 5g\sec(\theta)$, $T_2 = 5g\tan(\theta)$

$\rightarrow T_1 : T_2 = \sec(\theta) : \tan(\theta)$

So $T_1 > T_2$

$T_1\cos(\theta) = 5g$ (Resolve vertically)

$\rightarrow 98\cos(\theta) = 5g$

$\cos(\theta) = \frac{5g}{98} = \frac{1}{2}$

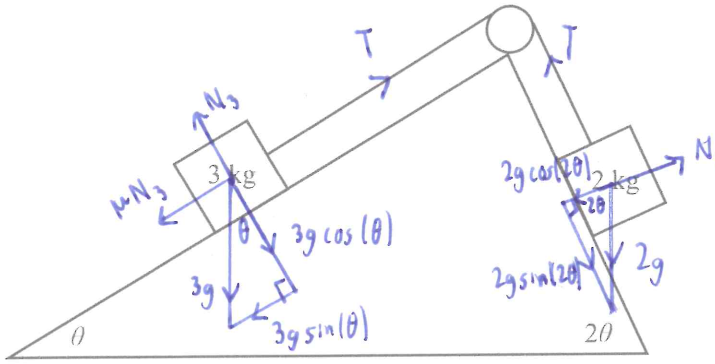
$\theta_{max} = \cos^{-1}\left(\frac{1}{2}\right)$

$= \boxed{\frac{\pi}{3}}$

· e.g. 2. VCAA 2011 Exam 2 Q4

Question 4

The diagram below shows particles of mass 3 kg and 2 kg on adjoining smooth planes with inclinations θ and 2θ respectively, where θ is measured in degrees and $\theta > 0$. The masses are connected by a light inextensible string passing over a smooth pulley. The tension in the string is T newtons, and the acceleration **up the plane** of the 3 kg mass is a m/sec^2.



a. For the 3 kg mass, write down an equation for its motion **up** the plane.

$\sum \underset{\sim}{F} = m\underset{\sim}{a} \rightarrow \underline{T - 3g\sin(\theta) = 3a} \quad \cdots ①$

1 mark

b. For the 2 kg mass, write down an equation for its motion **down** the other plane.

$2g\sin(2\theta) - T = 2a \quad \cdots ②$

1 mark

c. Show that $a = \frac{g\sin\theta}{5}(4\cos\theta - 3)$.

$① + ②: \quad 2g\sin(2\theta) - 3g\sin(\theta) = 5a$

$2g(2\sin(\theta)\cos(\theta)) - 3g\sin(\theta) = 5a$

$g\sin(\theta)(4\cos(\theta) - 3) = 5a$

$\therefore \boxed{a = \frac{g\sin(\theta)}{5}(4\cos(\theta) - 3)}$ (Shown)

2 marks

d. Find the angle θ for the system to be in equilibrium.

Express your answer in degrees, correct to the nearest tenth of a degree.

$a = 0 = \frac{g\sin(\theta)}{5}(4\cos(\theta) - 3)$

$\rightarrow 4\cos(\theta) - 3 = 0 \quad (\text{since } 0 < \theta < \frac{\pi}{2})$

$\theta = \cos^{-1}\left(\frac{3}{4}\right) = \boxed{41.4^0}$ (1 dp)

2 marks