Integral Calculus - Exercises

6.1 Antidifferentiation. The Indefinite Integral

In problems 1 through 7, find the indicated integral.

1. $\int \sqrt{x}dx$
   Solution.
   $$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx = \frac{2}{3}x^{\frac{3}{2}} + C = \frac{2}{3}x\sqrt{x} + C.$$

2. $\int 3e^x dx$
   Solution.
   $$\int 3e^x dx = 3\int e^x dx = 3e^x + C.$$

3. $\int (3x^2 - \sqrt{5x} + 2)dx$
   Solution.
   $$\begin{aligned}\int (3x^2 - \sqrt{5x} + 2)dx &= 3\int x^2 dx - \sqrt{5}\int \sqrt{x}dx + 2\int dx = \\ &= 3 \cdot \frac{1}{3}x^3 - \sqrt{5} \cdot \frac{2}{3}x\sqrt{x} + 2x + C = \\ &= x^3 - \frac{2}{3}x\sqrt{5x} + 2x + C.\end{aligned}$$

4. $\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right) dx$
   Solution.
   $$\begin{aligned}\int \left(\frac{1}{2x} - \frac{2}{x^2} + \frac{3}{\sqrt{x}}\right) dx &= \frac{1}{2}\int \frac{1}{x}dx - 2\int x^{-2}dx + 3\int x^{-\frac{1}{2}}dx = \\ &= \frac{1}{2}\ln|x| - 2 \cdot (-1)x^{-1} + 3 \cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{\ln|x|}{2} + \frac{2}{x} + 6\sqrt{x} + C.\end{aligned}$$

5. $\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx$
Solution.

$$\begin{aligned}\int \left(2e^x + \frac{6}{x} + \ln 2\right) dx &= 2\int e^x dx + 6\int \frac{1}{x} dx + \ln 2 \int dx = \\ &= 2e^x + 6\ln|x| + (\ln 2)x + C.\end{aligned}$$

6. $\int \frac{x^2+3x-2}{\sqrt{x}} dx$
Solution.

$$\begin{aligned}\int \frac{x^2+3x-2}{\sqrt{x}} dx &= \int x^{\frac{3}{2}} dx + 3\int x^{\frac{1}{2}} dx - 2\int x^{-\frac{1}{2}} dx = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 3 \cdot \frac{2}{3}x^{\frac{3}{2}} - 2 \cdot 2x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^{\frac{5}{2}} + 2x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + C = \\ &= \frac{2}{5}x^2\sqrt{x} + 2x\sqrt{x} - 4\sqrt{x} + C.\end{aligned}$$

7. $\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx$
Solution.

$$\begin{aligned}\int (x^3 - 2x^2)\left(\frac{1}{x} - 5\right) dx &= \int (x^2 - 5x^3 - 2x + 10x^2) dx = \\ &= \int (-5x^3 + 11x^2 - 2x) dx = \\ &= -5 \cdot \frac{1}{4}x^4 + 11 \cdot \frac{1}{3}x^3 - 2 \cdot \frac{1}{2}x^2 + C = \\ &= -\frac{5}{4}x^4 + \frac{11}{3}x^3 - x^2 + C.\end{aligned}$$

8. Find the function f whose tangent has slope $x^3 - \frac{2}{x^2} + 2$ for each value of x and whose graph passes through the point $(1, 3)$.
Solution. The slope of the tangent is the derivative of f. Thus

$$f'(x) = x^3 - \frac{2}{x^2} + 2$$

and so $f(x)$ is the indefinite integral

$$\begin{aligned}f(x) &= \int f'(x) dx = \int \left(x^3 - \frac{2}{x^2} + 2\right) dx = \\ &= \frac{1}{4}x^4 + \frac{2}{x} + 2x + C.\end{aligned}$$

Using the fact that the graph of f passes through the point $(1,3)$ you get

$$3 = \frac{1}{4} + 2 + 2 + C \quad \text{or} \quad C = -\frac{5}{4}.$$

Therefore, the desired function is $f(x) = \frac{1}{4}x^4 + \frac{2}{x} + 2x - \frac{5}{4}$.

9. It is estimated that t years from now the population of a certain lakeside community will be changing at the rate of $0.6t^2 + 0.2t + 0.5$ thousand people per year. Environmentalists have found that the level of pollution in the lake increases at the rate of approximately 5 units per 1000 people. By how much will the pollution in the lake increase during the next 2 years?

   Solution. Let $P(t)$ denote the population of the community t years from now. Then the rate of change of the population with respect to time is the derivative

   $$\frac{dP}{dt} = P'(t) = 0.6t^2 + 0.2t + 0.5.$$

   It follows that the population function $P(t)$ is an antiderivative of $0.6t^2 + 0.2t + 0.5$. That is,

   $$\begin{aligned} P(t) &= \int P'(t)dt = \int (0.6t^2 + 0.2t + 0.5)dt = \\ &= 0.2t^3 + 0.1t^2 + 0.5t + C \end{aligned}$$

   for some constant C. During the next 2 years, the population will grow on behalf of

   $$\begin{aligned} P(2) - P(0) &= 0.2 \cdot 2^3 + 0.1 \cdot 2^2 + 0.5 \cdot 2 + C - C = \\ &= 1.6 + 0.4 + 1 = 3 \text{ thousand people.} \end{aligned}$$

   Hence, the pollution in the lake will increase on behalf of $5 \cdot 3 = 15$ units.

10. An object is moving so that its speed after t minutes is $v(t) = 1+4t+3t^2$ meters per minute. How far does the object travel during 3rd minute?

    Solution. Let $s(t)$ denote the displacement of the car after t minutes. Since $v(t) = \frac{ds}{dt} = s'(t)$ it follows that

    $$s(t) = \int v(t)dt = \int (1 + 4t + 3t^2)dt = t + 2t^2 + t^3 + C.$$

    During the 3rd minute, the object travels

    $$\begin{aligned} s(3) - s(2) &= 3 + 2 \cdot 9 + 27 + C - 2 - 2 \cdot 4 - 8 - C = \\ &= 30 \text{ meters.} \end{aligned}$$

Homework

In problems 1 through 13, find the indicated integral. Check your answers by differentiation.

1.	$\int x^5 dx$	2.	$\int x^{\frac{3}{4}} dx$
3.	$\int \frac{1}{x^2} dx$	4.	$\int 5dx$
5.	$\int (x^{\frac{1}{2}} - 3x^{\frac{2}{3}} + 6)dx$	6.	$\int \left(3\sqrt{x} - \frac{2}{x^3} + \frac{1}{x}\right) dx$
7.	$\int \left(\frac{e^x}{2} + x\sqrt{x}\right) dx$	8.	$\int \left(\sqrt{x^3} - \frac{1}{2\sqrt{x}} + \sqrt{2}\right) dx$
9.	$\int \left(\frac{1}{3x} - \frac{3}{2x^2} + e^2 + \frac{\sqrt{x}}{2}\right) dx$	10.	$\int \frac{x^2+2x+1}{x^2} dx$
11.	$\int x^3 \left(2x + \frac{1}{x}\right) dx$	12.	$\int \sqrt{x}(x^2 - 1)dx$
13.	$\int x(2x + 1)^2 dx$		

14. Find the function whose tangent has slope $4x + 1$ for each value of x and whose graph passes through the point $(1, 2)$.

15. Find the function whose tangent has slope $3x^2 + 6x - 2$ for each value of x and whose graph passes through the point $(0, 6)$.

16. Find a function whose graph has a relative minimum when $x = 1$ and a relative maximum when $x = 4$.

17. It is estimated that t months from now the population of a certain town will be changing at the rate of $4 + 5t^{\frac{2}{3}}$ people per month. If the current population is 10000, what will the population be 8 months from now?

18. An environmental study of a certain community suggests that t years from now the level of carbon monoxide in the air will be changing at the rate of $0.1t + 0.1$ parts per million per year. If the current level of carbon monoxide in the air is 3.4 parts per million, what will the level be 3 years from now?

19. After its brakes are applied, a certain car decelerates at the constant rate of 6 meters per second per second. If the car is traveling at 108 kilometers per hour when the brakes are applied, how far does it travel before coming to a complete stop? (Note: 108 kmph is the same as 30 mps.)

20. Suppose a certain car supplies a constant deceleration of A meters per second per second. If it is traveling at 90 kilometers per hour (25 meters per second) when the brakes are applied, its stopping distance is 50 meters.

    (a) What is A?

(b) What would the stopping distance have been if the car had been traveling at only 54 kilometers per hour when the brakes were applied?

(c) At what speed is the car traveling when the brakes are applied if the stopping distance is 56 meters?

Results.

1.	$\frac{1}{6}x^6 + C$	2.	$\frac{4}{7}x^{\frac{7}{4}} + C$
3.	$-\frac{1}{x} + C$	4.	$5x + C$
5.	$\frac{2}{3}x^{\frac{3}{2}} - \frac{9}{5}x^{\frac{5}{3}} + 6x + C$	6.	$2x^{\frac{3}{2}} + \frac{1}{x^2} + \ln\lvert x\rvert + C$
7.	$\frac{1}{2}e^x + \frac{2}{5}x^{\frac{5}{2}} + C$	8.	$\frac{2}{5}\sqrt{(x^3)}x - \sqrt{x} + \sqrt{2}x + C$
9.	$\frac{1}{3}\ln\lvert x\rvert + \frac{3}{2x} + e^2x + \frac{1}{3}x^{\frac{3}{2}} + C$	10.	$x - \frac{1}{x} + 2\ln x + C$
11.	$\frac{2}{5}x^5 + \frac{1}{3}x^3 + C$	12.	$\frac{2}{7}x^{\frac{7}{2}} - \frac{2}{3}x^{\frac{3}{2}} + C$
13.	$x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 + C$		

14. $f(x) = 2x^2 + x - 1$

15. $f(x) = x^3 + 3x^2 - 2x + 6$

16. $f(x) = \frac{1}{3}x^3 - \frac{5}{2}x^2 + 4x$; not unique

17. 10128

18. 4.15 parts per million

19. 75 meters

20. (a) $A = 6.25$
    (b) 42 meters
    (c) 120.37 kilometers per hour