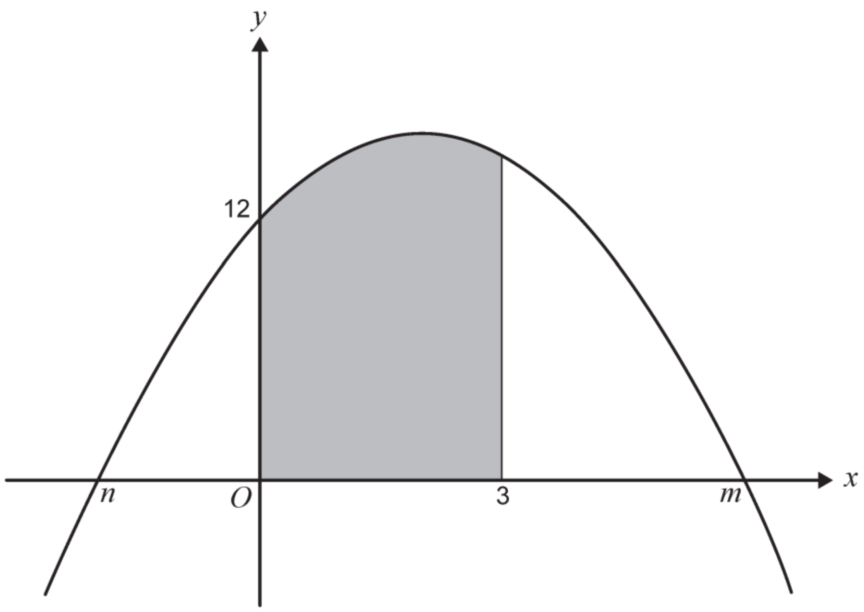
UNIT 3 & 4 MATHEMATICAL METHODS
VCAA EXAMINATION 1 – 2006 TO 2017

ANTIDIFFERENTIATION – AREAS UNDER CURVES

QUESTION 1 – 2006 (HARD)

Part of the graph of the function $f: R \to R$, $f(x) = -x^2 + ax + 12$ is shown below. If the shaded area is 45 square units, find the values of a, m and n where m and n are the x-axis intercepts of the graph of $y = f(x)$.



5 marks

QUESTION 2 – 2007 (HARD)

The area of the region bounded by the curve with equation $y = kx^{\frac{1}{2}}$, where k is a positive constant, the x-axis and the line with equation $x = 9$ is 27. Find k.

3 marks

QUESTION 3 – 2008 (HARD)

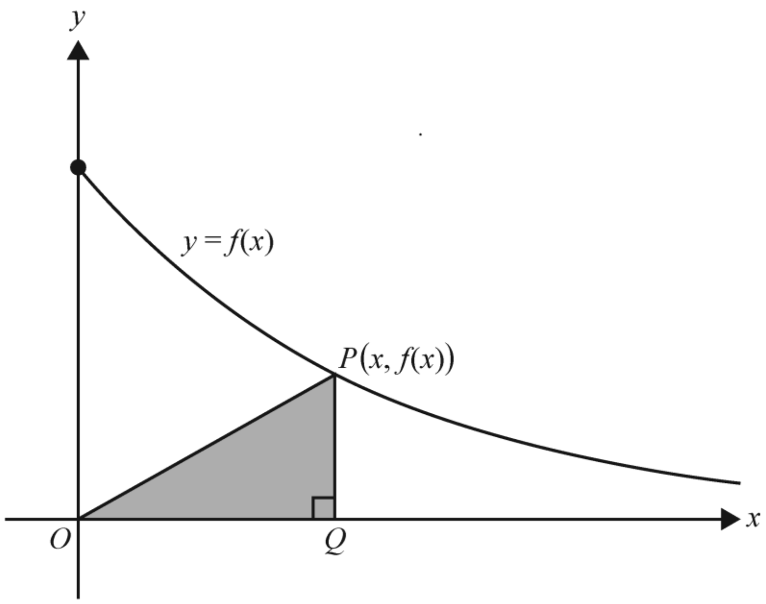
The area of the region bounded by the y-axis, the x-axis, the curve $y = e^{2x}$ and the line $x = C$, where C is a positive real constant, is $\frac{5}{2}$. Find C.

3 marks

QUESTION 4 – 2013 (AVERAGE – HARD – HARD)

Let $f: [0, \infty) \to R, f(x) = 2e^{-\frac{x}{5}}$.

A right-angled triangle OQP has vertex O at the origin, vertex Q on the x-axis and vertex P on the graph of f, as shown. The coordinates of P are $(x, f(x))$.

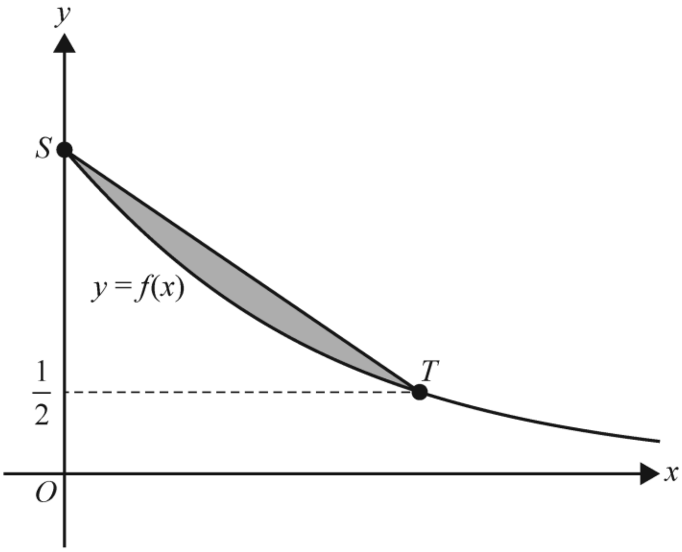


a. Find the area, A, of the triangle OQP in terms of x. 1 mark

b. Find the maximum area of triangle *OQP* and the value of *x* for which the maximum occurs. 3 marks

c. Let S be the point on the graph of f on the y-axis and let T be the point on the graph of f with the y-coordinate $\frac{1}{2}$.

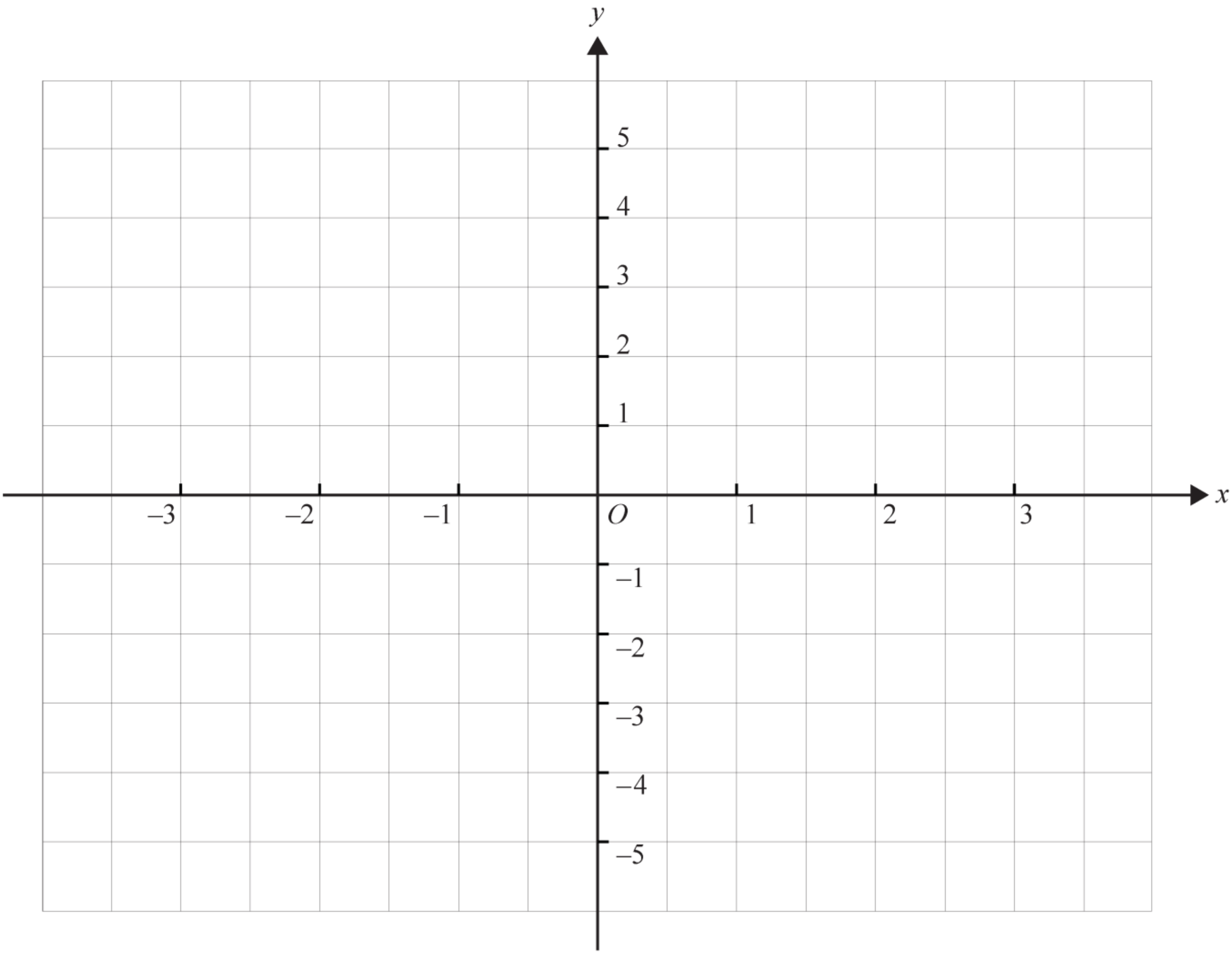
Find the area of the region bounded by the graph of f and the line segment ST. 3 marks



QUESTION 5 – 2016 (AVERAGE – HARD)

Let $f: R\backslash\{1\} \to R$, where $f(x) = 2 + \dfrac{3}{x-1}$. ☆

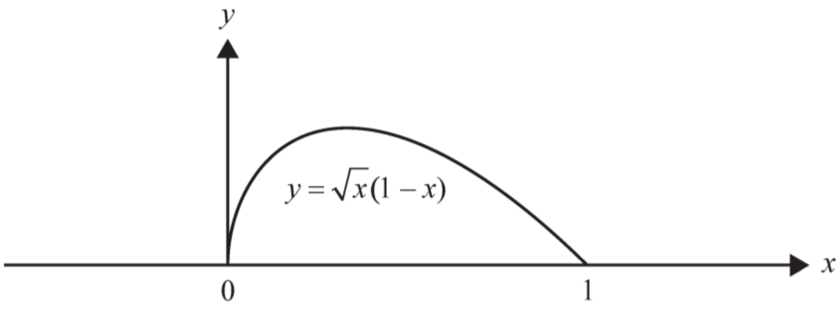
a. Sketch the graph of f. Label the axis intercepts with their coordinates and label any asymptotes with the appropriate equation. 3 marks



b. Find the area enclosed by the graph of f, the lines $x = 2$ and $x = 4$, and the x-axis. 2 marks

QUESTION 6 – 2017 (HARD – HARD – HARD – HARD)

The graph of $f:[0,\ 1]\to R,\ f(x)=\sqrt{x}(1-x)$ is shown below.

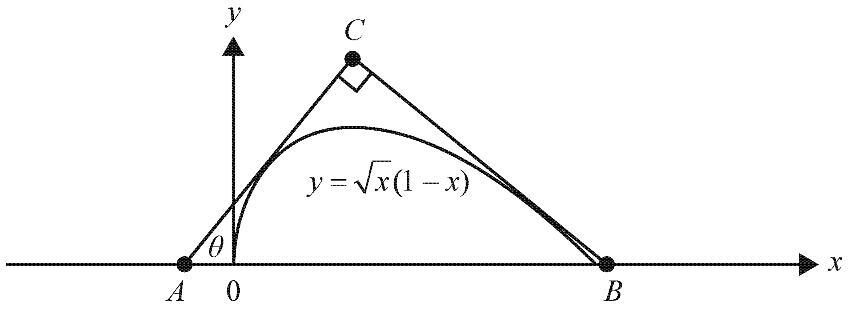


a. Calculate the area between the graph of f and the x-axis. 2 marks

b. For x in the interval (0, 1), show that the gradient of the tangent to the graph of f is $\frac{1-3x}{2\sqrt{x}}$. 1 mark

The edges of the **right-angled** triangle *ABC* are the line segments *AC* and *BC*, which are tangent to the graph of *f*, and the line segment *AB*, which is part of the horizontal axis, as shown below.

Let θ be the angle that *AC* makes with the positive direction of the horizontal axis, where $45° \le \theta < 90°$.



c. Find the equation of the line through *B* and *C* in the form $y = mx + c$, for $\theta = 45°$. 2 marks

d. Find the coordinates of *C* when $\theta = 45°$. 4 marks

SOLUTIONS

QUESTION 1

Shaded Area $= \int_0^3 (-x^2 + ax + 12)\, dx = 45$

$$\Rightarrow \left[-\frac{x^3}{3} + \frac{ax^2}{2} + 12x\right]_0^3 = 45 \quad (M1)$$

$$\left(-9 + \frac{9a}{2} + 36\right) - (0) = 45$$

$$\frac{9a}{2} + 27 = 45$$

$$\frac{9a}{2} = 18$$

$$a = 4 \quad (A1)$$

$\therefore\ f(x) = -x^2 + 4x + 12$

X int, Let $y = 0$: $\quad -x^2 + 4x + 12 = 0 \quad (M1)$

$$x^2 - 4x - 12 = 0$$

$$(x - 6)(x + 2) = 0$$

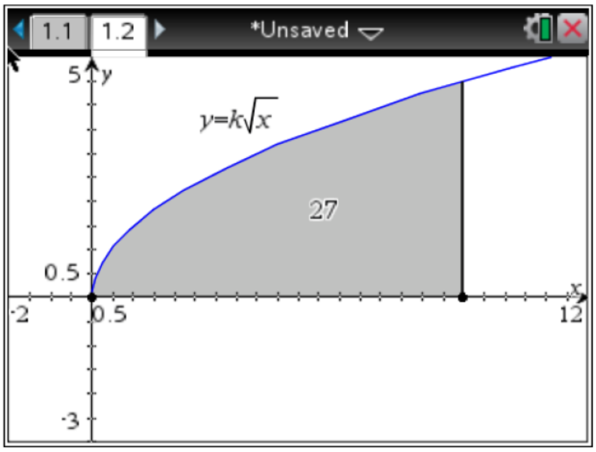
$$x = -2 \text{ or } 6$$

$a = 4$, $n = -2$, $m = 6$
(A1) (A1)

Difficulty: 16% of students answered this question correctly
Average Score: 0.2 out of 1

QUESTION 2

$y = kx^{\frac{1}{2}}$



$\text{Area } = \int_0^9 kx^{\frac{1}{2}} dx$ **(M1)**

$\int_0^9 kx^{\frac{1}{2}} dx = 27$

$\therefore k\left[\dfrac{x^{\frac{3}{2}}}{3/2}\right]_0^9 = 27$

$\dfrac{2k}{3}\left[9^{\frac{3}{2}}\right] = 27$

(M1)

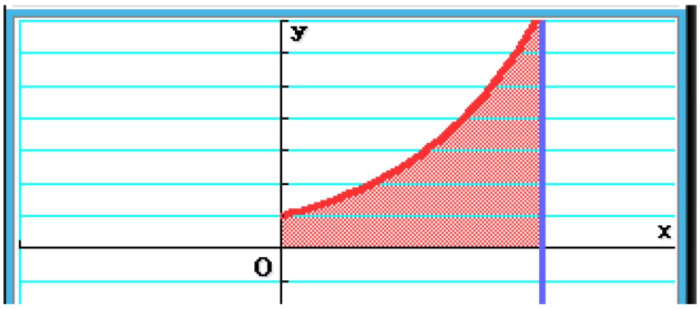
$\therefore \dfrac{2k}{3} \times 27 = 27$

$\dfrac{2k}{3} = 1 \therefore k = \dfrac{3}{2}$ **(A1)**

Difficulty: 33% of students answered this question correctly
Average Score: 1.6 out of 3

QUESTION 3

$y = e^{2x}$ and $x = C$



$x = C$

Area $= \frac{5}{2} = \int_0^C e^{2x} dx$ **(M1)**

$= \frac{1}{2}\left[e^{2x}\right]_0^C$

$= \frac{1}{2}\left(e^{2C} - e^0\right)$

Where Area $= \frac{5}{2}$

Giving $\frac{1}{2}\left(e^{2C} - 1\right) = \frac{5}{2}$ **(M1)**

$\therefore e^{2C} - 1 = 5$

$e^{2C} = 6$

$\therefore C = \frac{1}{2}\log_e 6$ **(A1)**

Alternative answer: $C = \log_e \sqrt{6}$

Difficulty: 47% of students answered this question correctly
Average Score: 1.9 out of 3

QUESTION 4

a.

$$\text{Area} = \frac{1}{2} \times b \times h = \frac{1}{2} \times x \times f(x)$$

$$= xe^{-\frac{x}{5}} \text{ square units} \quad (A1)$$

Difficulty: 55% of students answered this question correctly
Average Score: 0.6 out of 1

b.

$$A = xe^{-x/5}$$

$$\frac{dA}{dx} = x\left(-\frac{1}{5}e^{-x/5}\right) + \left(e^{-x/5}\right) \quad (M1)$$

$$= e^{-x/5}\left(-\frac{x}{6} + 1\right)$$

$$= e^{-x/5}\left(1 - \frac{x}{5}\right)$$

$$\text{Let } \frac{dA}{dx} = 0 : \quad e^{-x/5}\left(1 - \frac{x}{5}\right) = 0$$

$$1 - \frac{x}{5} = 0$$

$$x = 5 \quad (A1)$$

$$A_{max} = xe^{-x/5} = 5e^{-1} \text{ square units} \quad (A1)$$

Difficulty: 26% of students answered this question correctly
Average Score: 1.2 out of 3

c.

Upper limit: $y = 2e^{-x/5}$

$$\frac{1}{2} = 2e^{-x/5}$$

$$\frac{1}{4} = e^{-x/5}$$

$$\log_e\left(\frac{1}{4}\right) = \log_e e^{-x/5}$$

$$\log_e\left(\frac{1}{4}\right) = -\frac{x}{5}$$

$$5\log_e\left(\frac{1}{4}\right) = -x$$

$$\therefore x = 5\log_e 4 \quad (\text{M1})$$

Area Shaded Region: $\frac{1}{2}\left(2+\frac{1}{2}\right)\times 5\log_e 4 - \int_0^{5\log_e 4} 2e^{-x/5}\,dx \quad (\text{M1})$

$$= \frac{25}{2}\log_e 2 - \left[-10e^{-x/5}\right]^{5\log_e 4}$$

$$= \frac{25}{2}\log_e 2 - \frac{15}{2}$$

$$= \frac{5}{2}\left(5\log_e 2 - 3\right) \text{ units}^2 \quad (\text{A1})$$

Difficulty: 7% of students answered this question correctly
Average Score: 1 out of 3

QUESTION 5

a.

$f(x) = 2 + \frac{3}{x-1}$

x int, Let $y = 0$:

$2 + \frac{3}{x-1} = 0$

$\frac{3}{x-1} = -2$

$3 = -2(x-1)$

$3 = -2x + 2$

$2x = -1$

$x = -\frac{1}{2}$ $\therefore \left(-\frac{1}{2}, 0\right)$ $\left(A\frac{1}{2}\right)$

Y int, Let $x = 0$:

$2 + \frac{3}{-1} = 2 - 3 = -1$

$\therefore (0, -1)$ $\left(A\frac{1}{2}\right)$

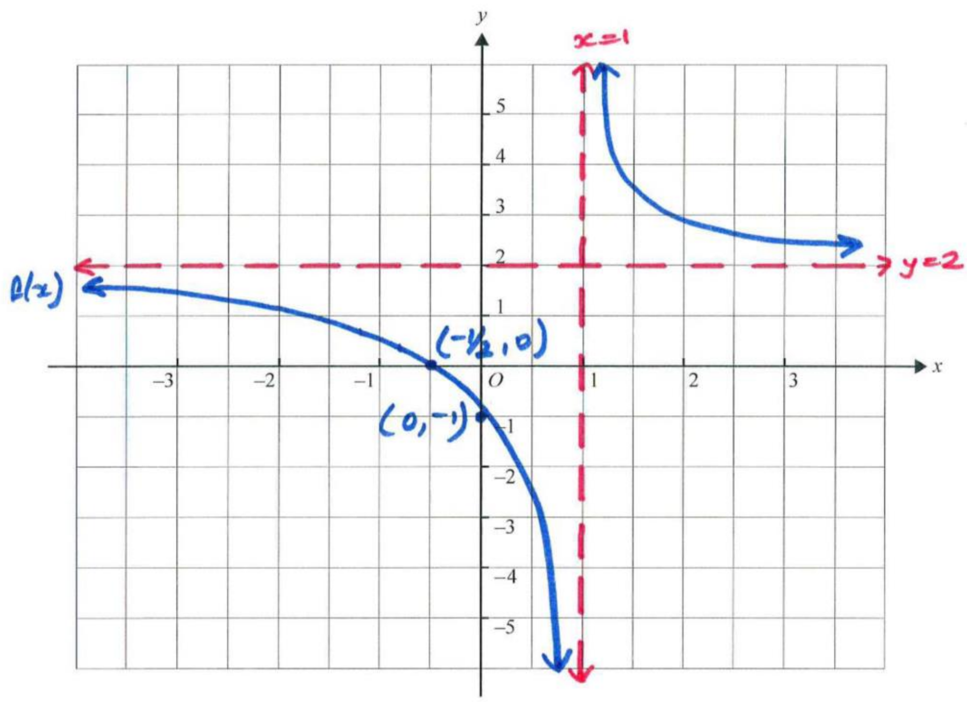
x asy: Let $x - 1 = 0$

$\therefore x = 1$ $\left(A\frac{1}{2}\right)$

Y asy: $y = 2$ $\left(A\frac{1}{2}\right)$

Shape = A1

Round down



Difficulty: 57% of students answered this question correctly
Average Score: 2.3 out of 3

b.

$$\text{Area} = \int_2^4 \left(2 + \frac{3}{x-1}\right) dx = \left[2x + 3\log_e(x-1)\right]_2^4 \quad (M1)$$

$$= \left(8 + 3\log_e 3\right) - \left(4 + 3\log_e 1\right)$$

$$= 4 + 3\log_e 3 \quad (A1)$$

Difficulty: 46% of students answered this question correctly
Average Score: 1.2 out of 2

QUESTION 6

a.

$$y = \sqrt{x}(1-x) = x^{1/2} - x^{3/2}$$

$$\text{Area} = \int_0^1 \left(x^{1/2} - x^{3/2}\right) dx$$

$$= \left[\frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5}\right]_0^1 \quad (M1)$$

$$= \frac{2}{3} - \frac{2}{5}$$

$$= \frac{4}{15} \quad (A1)$$

Difficulty: 47% of students answered this question correctly
Average Score: 0.6 out of 2

b.

$$y = x^{1/2} - x^{3/2}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{3}{2}x^{\frac{1}{2}}$$

$$= \frac{1}{2\sqrt{x}} - \frac{3\sqrt{x}}{2}$$

$$= \frac{1 - 3\sqrt{x}\sqrt{x}}{2\sqrt{x}}$$

$$= \frac{1-3x}{2\sqrt{x}} \quad (A1)$$

Difficulty: 35% of students answered this question correctly
Average Score: 0.3 out of 1

c.

$$m_{AC} = \tan 45^\circ = 1$$

$$\therefore m_{BC} = -1$$

$$m_{tangent} = \frac{1-3x}{2\sqrt{x}}$$

$$\text{Let } \frac{1-3x}{2\sqrt{x}} = -1 \quad (M1)$$

$$1 - 3x = -2\sqrt{x}$$

$$x = 1$$

$$\text{When } x = 1,\ y = \sqrt{1}(1-1) = 0 \quad \therefore (1, 0)$$

$$\text{Tangent: } y - y_1 = m(x - x_1)$$

$$y - 0 = -1(x-1)$$

$$y = 1 - x \quad (A1)$$

Difficulty: 17% of students answered this question correctly
Average Score: 0.2 out of 2

d.

Coordinates of C = Point of Intersection of Line AC and Line BC

Line BC: $y = -x + 1$

Line AC: $y - y_1 = m(x - x_1)$

$m_{AC} = 1$

Let $\frac{1-3x}{2\sqrt{x}} = 1$ (M1)

$1 - 3x = 2\sqrt{x}$

$3x + 2\sqrt{x} - 1 = 0$

$(3\sqrt{x} - 1)(\sqrt{x} + 1) = 0$

$\therefore 3\sqrt{x} - 1 = 0$

$\sqrt{x} = \frac{1}{3}$

$x = \frac{1}{9}$

$y - \frac{8}{27} = 1\left(x - \frac{1}{9}\right)$

$y = x - \frac{1}{9} + \frac{8}{27}$

$y = x - \frac{3}{27} + \frac{8}{27}$

$y = x + \frac{5}{27}$

when $x = \frac{1}{9}$, $y = \frac{1}{3}\left(1 - \frac{1}{9}\right) = \frac{8}{27}$ $\therefore \left(\frac{1}{9}, \frac{8}{27}\right)$

Let Line AC = Line BC

$x + \frac{5}{27} = -x + 1$ (M1)

$2x = 1 - \frac{5}{27}$

$2x = \frac{22}{27}$

$x = \frac{11}{27}$ $\therefore y = \frac{16}{27}$ $\therefore C = \left(\frac{11}{27}, \frac{16}{27}\right)$

Difficulty: 9% of students answered this question correctly
Average Score: 0.2 out of 4