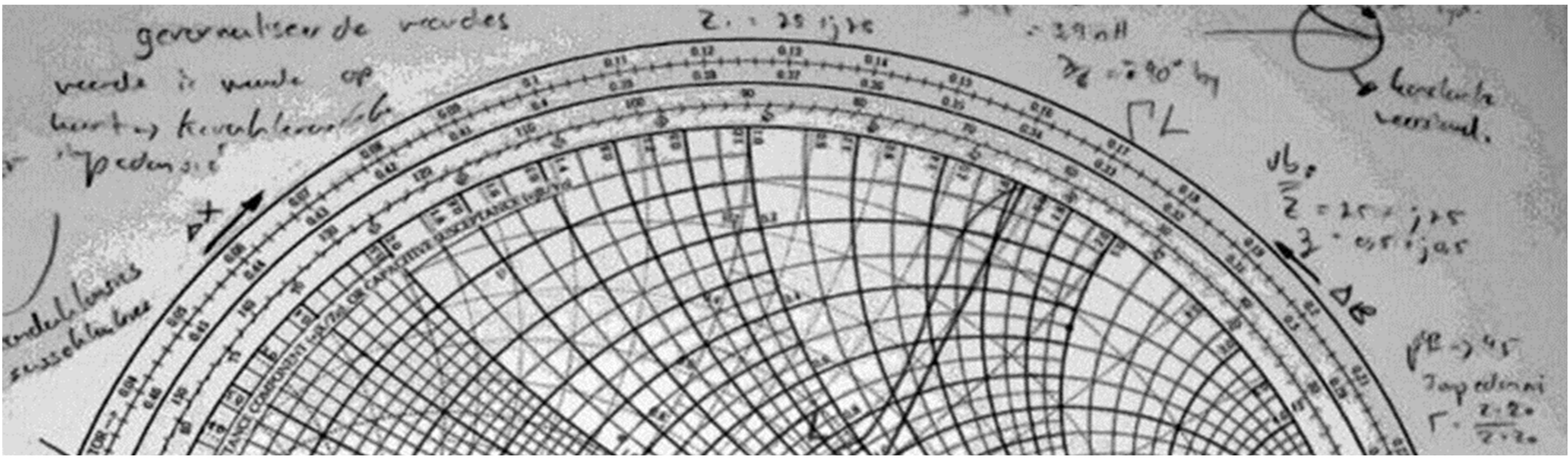
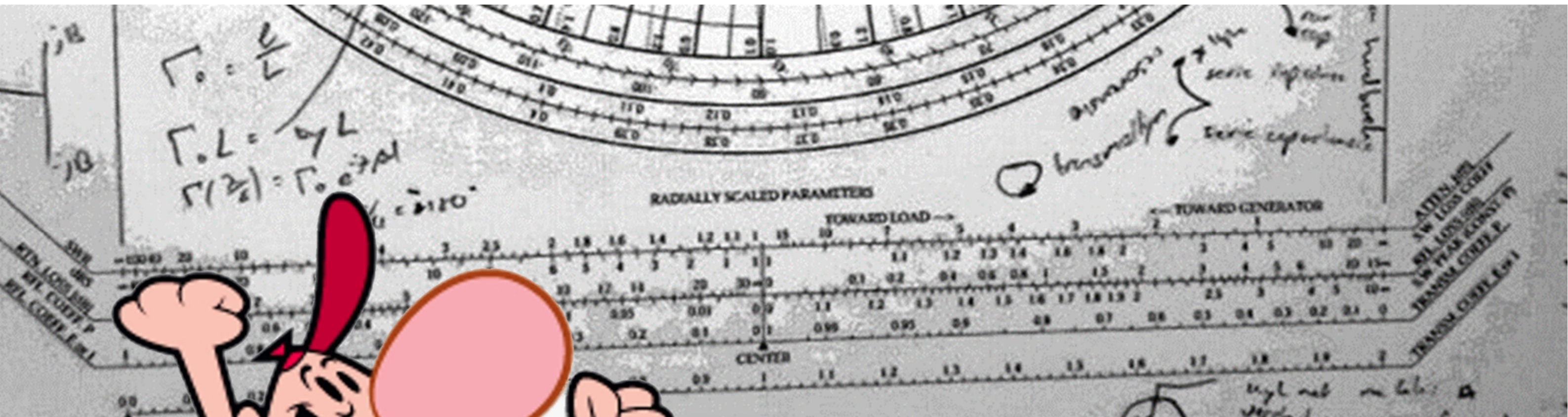


UNIT 3 SPECIALIST MATHS

DIFFERENTIATION TECHNIQUES & RATIONAL FUNCTIONS

REVISION NOTES FOR YOUR SACS & EXAMS



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Chapter 6 Differentiation and Rational Functions

Exercise 6A Differentiation

→ Review (also from Mathematical Methods Units 3 and 4)

Calc. Menu → Calculus(4) → Derivative(1)

$F(x)$, y	$F'(x)$, $\frac{dy}{dx}$, $\frac{d}{dx}[F(x)]$
C (constant)	0
x^n	nx^{n-1}
$x^{f(x)}$	$f'(x) \times x^{f(x)}$
$e^{f(x)}$	$f'(x)\, e^{f(x)}$
$\log_e \lvert f(x) \rvert$	$\frac{f'(x)}{f(x)}$
$\sin(f(x))$	$f'(x)\cos[f(x)]$
$\cos(f(x))$	$-f'(x)\sin[f(x)]$
$\tan(f(x))$	$f'(x)\sec^2[f(x)]$
$\cot(f(x))$	$-f'(x)\,\mathrm{cosec}^2[f(x)]$
$\sec(f(x))$	$f'(x)\tan[f(x)]\sec[f(x)]$
$\mathrm{cosec}(f(x))$	$-f'(x)\cot[f(x)]\,\mathrm{cosec}[f(x)]$

→ Product Rule: $y = uv \rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$f(x) = g(x)h(x) \rightarrow f'(x) = g'(x)h(x) + g(x)h'(x)$$

→ Quotient Rule: $y = \frac{u}{v} \rightarrow \frac{dy}{dx} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2}$

$$f(x) = \frac{g(x)}{h(x)} \rightarrow f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{(h(x))^2}$$

→ Chain Rule: If $y = h(u)$ and $u = g(x)$, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$f(x) = h(g(x)) \rightarrow f'(x) = h'(g(x)) \cdot g'(x)$$

→ Differentiation examples.

e.g.1. Differentiate $y = \frac{\sqrt{x}}{\tan(x)}$ with respect to x.

$$\rightarrow \frac{dy}{dx} = \frac{\tan(x) \times \frac{1}{2}x^{-1/2} - \sqrt{x}\sec^2(x)}{\tan^2(x)}$$

$$= \frac{1}{2\sqrt{x}\tan(x)} - \frac{\sqrt{x}}{\cos^2(x)} \times \frac{\cos^2(x)}{\sin^2(x)}$$

$$= \frac{\sqrt{x}}{2x\tan(x)} - \frac{\sqrt{x}}{\sin^2(x)} = \boxed{\sqrt{x}\left(\frac{\cot(x)}{2x} - \mathrm{cosec}^2(x)\right)}$$

e.g.2. Find the derivative of $\log_e|\mathrm{cosec}(x) - \cot(x)|$ with respect to x.

$$\rightarrow \frac{d}{dx}(\mathrm{cosec}(x) - \cot(x)) = \frac{d}{dx}\left(\frac{1}{\sin(x)} - \frac{\cos(x)}{\sin(x)}\right)$$

$$= -(\sin(x))^{-2}\cos(x) - \frac{\sin(x)(-\sin(x)) - \cos^2(x)}{\sin^2(x)}$$

$$= -\frac{\cos(x)}{\sin(x)} \times \frac{1}{\sin(x)} - \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)}$$

$$= -\cot(x)\,\mathrm{cosec}(x) - \mathrm{cosec}^2(x) = \mathrm{cosec}(x)(\mathrm{cosec}(x) - \cot(x))$$

Let $y = \log_e|\mathrm{cosec}(x) - \cot(x)|$. Then $\frac{dy}{dx} = \frac{\mathrm{cosec}(x)(\mathrm{cosec}(x) - \cot(x))}{\mathrm{cosec}(x) - \cot(x)} = \boxed{\mathrm{cosec}(x)}$

· e.g.3. Let $f(x) = \sec^3\left(\frac{x}{4}+3\sqrt{x}\right)$. Find $f'(x)$.

$\rightarrow f(x) = \left[\cos\left(\frac{x}{4}+3\sqrt{x}\right)\right]^{-3}$, $\cos\left(\frac{x}{4}+3\sqrt{x}\right) \neq 0$

$f'(x) = -3\left(\cos\left(\frac{x}{4}+3\sqrt{x}\right)\right)^{-4} \times -\left(\frac{1}{4}+\frac{3}{2\sqrt{x}}\right)\sin\left(\frac{x}{4}+3\sqrt{x}\right)$

$= 3\left(\frac{1}{4}+\frac{3}{2\sqrt{x}}\right)\sin\left(\frac{x}{4}+3\sqrt{x}\right)\sec^4\left(\frac{x}{4}+3\sqrt{x}\right)$

$\therefore = \frac{3(\sqrt{x}+6)}{4\sqrt{x}}\sin\left(\frac{x}{4}+3\sqrt{x}\right)\sec^4\left(\frac{x}{4}+3\sqrt{x}\right)$

· e.g.4. Find $\frac{dy}{dx}$ if $y = x^x$.

$y = e^{\log_e(x^x)} = e^{x\ln(x)}$

Let $u = x\ln(x)$ and $y = e^u$.

$\frac{du}{dx} = x \times \frac{1}{x} + \ln(x)$ $\quad \frac{dy}{du} = e^u$

$= 1+\ln(x)$

$\frac{dy}{dx} = (1+\ln(x)) \cdot e^u$

$= (1+\ln(x)) \cdot e^{x\ln(x)}$

$\therefore = \boxed{(1+\ln(x)) \cdot x^x}$

Exercise 6B Derivatives of $x = f(y)$

$\Rightarrow \frac{dy}{dx} = \frac{1}{dx/dy}$, $\frac{dx}{dy} \neq 0$

· e.g.1 Find the gradient of the curve $x = ye^y$ at $x = e$

$\frac{dx}{dy} = ye^y + e^y = e^y(y+1)$

$\frac{dy}{dx} = \frac{1}{e^y(y+1)}$, $y \neq -1$ ··· ①

When $x = e$, $e = ye^y \rightarrow y = 1$

Sub $y = 1$ into ①, $\frac{dy}{dx} = \boxed{\frac{1}{2e}}$

· e.g.2. Find the gradient of the curve $x = 2y^2 - 4y$ at $x = -2$.

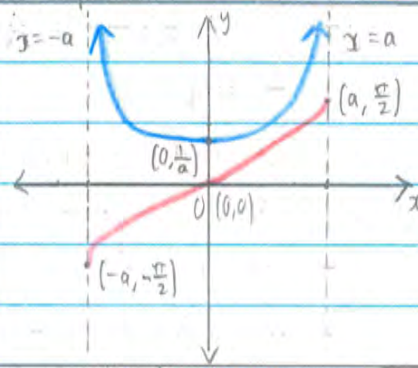
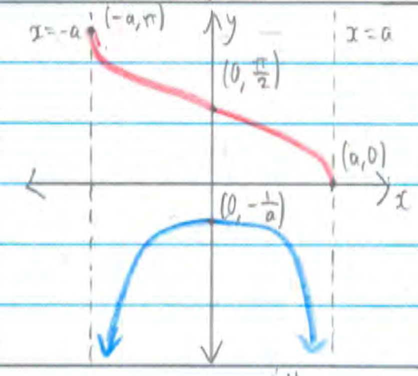
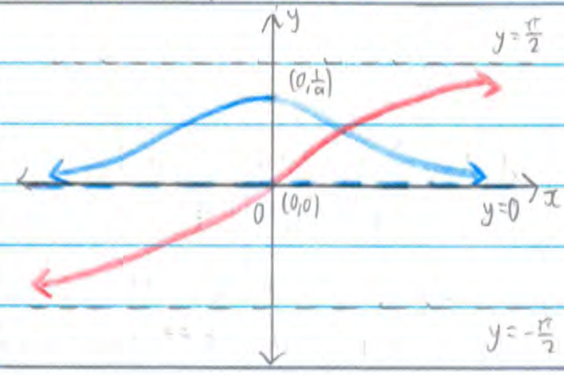
$\frac{dx}{dy} = 4y - 4 \rightarrow \frac{dy}{dx} = \frac{1}{4(y-1)}$, $y \neq 1$

When $x = -2$, $-2 = 2y^2 - 4y$

$2y^2 - 4y + 2 = 0 \rightarrow 2(y-1)^2 = 0 \rightarrow y = 1$

Sub $y = 1$ into $\frac{dy}{dx} \rightarrow$ $\boxed{\text{undefined}}$

Exercise 6C Derivatives of inverse circular functions (chain rule)

$f(x)$	$f'(x)$	Graph of $f'(x)$
$f: [-a,a] \to R$, $f(x) = \sin^{-1}\left(\frac{x}{a}\right)$	$\frac{1}{\sqrt{a^2-x^2}}$	
$f: [-a,a] \to R$, $f(x) = \cos^{-1}\left(\frac{x}{a}\right)$	$\frac{-1}{\sqrt{a^2-x^2}}$	
$f: R \to R$, $f(x) = \tan^{-1}\left(\frac{x}{a}\right)$	$\frac{a}{a^2+x^2}$	

- e.g.1. Find the derivative of $\arcsin(1-2x)$. Give your answer in the form $\frac{a}{\sqrt{g(x)-x^2}}$ where a is a constant and $g(x)$ is a function.

  Let $y = \arcsin(1-2\sqrt{x})$ and $u = 1-2x^{1/2}$. Then $y = \arcsin(u)$

  $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{\sqrt{1-u^2}} \times \left(-\frac{1}{\sqrt{x}}\right)$$

  $$= \frac{-1}{\sqrt{x}\sqrt{1-(1-2\sqrt{x})^2}} = \frac{-1}{\sqrt{x}\sqrt{4\sqrt{x}-4x}}$$

  $$= \boxed{\frac{-1}{2\sqrt{x\sqrt{x}-x^2}}}$$

- e.g.2. (a) Find the derivative of $\arccos\left(\frac{1}{3-2x}\right)$. Give your answer in the form $\frac{-1}{|ax-b|\sqrt{x^2-bx+a}}$, where a and b are real constants.

  Let $y = \arccos\left(\frac{1}{3-2x}\right)$ and $u = \frac{1}{3-2x}$. Then $y = \arccos(u)$

  $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{-1}{\sqrt{1-u^2}} \times \frac{2}{(3-2x)^2}$$

  Continued next page...

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-\frac{1}{(3-2x)^2}}} \times \frac{2}{(3-2x)^2} = \frac{-1}{\sqrt{\frac{(3-2x)^2-1}{(3-2x)^2}}} \times \frac{2}{(3-2x)^2}$$ (N.B. $\sqrt{x^2} = |x|$ $\rightarrow$ Modulus)

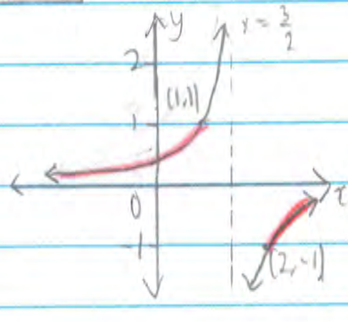
$$= \frac{-|3-2x|}{\sqrt{(3-2x)^2-1}} \times \frac{2}{|3-2x|^2} = \frac{-1}{\sqrt{8-12x+4x^2}} \times \frac{\sqrt{4}}{|3-2x|}$$

$$= \boxed{\frac{-1}{|2x-3|\sqrt{x^2-3x+2}}}$$

(b) Find the implied domain and range of $\arccos\left(\frac{1}{3-2x}\right)$.

$\rightarrow$ Domain: Solve $-1 \leq \frac{1}{3-2x} \leq 1 \rightarrow \boxed{x \in (-\infty, 1] \cup [2, +\infty)}$

- $1 = \frac{1}{3-2x} \rightarrow 3-2x = 1 \rightarrow x = 1$
- $-1 = \frac{1}{3-2x} \rightarrow 3-2x = -1 \rightarrow x = 2$



$\rightarrow$ Range: $\frac{dy}{dx} < 0$ over $x \in \mathbb{R} \rightarrow$ decreasing function

- Values of $y = f(x)$ for $x \in (-\infty, 1]$
  - $x = 1$: $y = f(1) = \arccos(1) = 0$
  - $x \rightarrow -\infty$: $y \rightarrow f(0) = \arccos(0) = \frac{\pi}{2}$
- Values of $y = f(x)$ for $x \in [2, \infty)$
  - $x = 2$: $y = f(-1) = \arccos(-1) = \pi$
  - $x \Rightarrow +\infty$: $y = f(0) = \arccos(0) = \frac{\pi}{2}$

$\therefore$ Range: $\boxed{y \in [0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]}$

- e.g. 3. Find the derivative of $\tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right)$. Hence, draw a graph of $y = \tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right)$, labelling all endpoints with coordinates and all asymptotes with their equation.

$\rightarrow$ Let $y = \tan^{-1}\left(\frac{2}{\sqrt{x+1}}\right)$, and $u = 2(x+1)^{-1/2}$. Then $y = \tan^{-1}(u)$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{1+u^2} \times \left(-(x+1)^{-3/2}\right)$$

$$= \frac{-1}{\left(\left(\frac{2}{\sqrt{x+1}}\right)^2 + 1\right)(x+1)^{3/2}} = \frac{-1}{\left(\frac{4}{x+1} + 1\right)(x+1)^{3/2}}$$

$$= \frac{-(x+1)}{(x+5)(x+1)^{3/2}} = \boxed{\frac{-1}{(x+5)\sqrt{x+1}}}$$

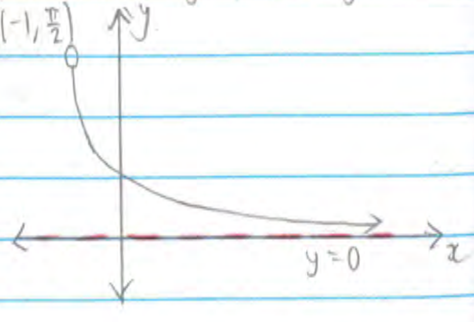
$\rightarrow$ Asymptotes: Undefined when $x = -1$ (but not a vertical asymptote!)

$x \rightarrow -1^+$: $y \rightarrow \tan^{-1}(+\infty) = \frac{\pi}{2} \rightarrow$ Open endpoint at $(-1, \frac{\pi}{2})$

$x \rightarrow +\infty$: $y = \tan^{-1}(0) = 0 \rightarrow$ Horizontal asymptote: $y = 0$

$\rightarrow$ Shape: $\frac{dy}{dx} < 0 \rightarrow$ decreasing function

Domain: $x > -1$



First derivative rules (General)

- $\frac{d}{dx}\left(\sin^{-1}(f(x))\right) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$, $-1 < f(x) < 1$
- $\frac{d}{dx}\left(\cos^{-1}(f(x))\right) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$, $-1 < f(x) < 1$
- $\frac{d}{dx}\left(\tan^{-1}(f(x))\right) = \frac{f'(x)}{1+(f(x))^2}$, $x \in \mathbb{R}$

Exercise 6D Second derivatives

→ Second derivative → 'derivative of the derivative'

- 'gradient of the gradient', or 'rate of change of gradient'
- Notation → $\frac{d^2y}{dx^2} = y'' = f''(x)$ (pronounced d-2-y d-x-squared)
- N.B. may not exist at a point even if first derivative does
  - e.g. $f(x) = x^{4/3}$, $f'(x) = \frac{4}{3}x^{1/3}$, $f''(x) = \frac{4}{9}x^{-2/3}$

→ Derivative examples

- e.g. 1. Find $f''(x)$ if $f(x) = \operatorname{cosec}\left(\frac{x}{4}\right) = \left(\sin\left(\frac{x}{4}\right)\right)^{-1}$

$\rightarrow f'(x) = -\left(\sin\left(\frac{x}{4}\right)\right)^{-2}\left[\frac{1}{4}\cos\left(\frac{x}{4}\right)\right] = -\frac{1}{4}\cos\left(\frac{x}{4}\right)\left(\sin\left(\frac{x}{4}\right)\right)^{-2}$

Let $u = -\frac{1}{4}\cos\left(\frac{x}{4}\right)$ and $v = \left(\sin\left(\frac{x}{4}\right)\right)^{-2}$

$\frac{du}{dx} = \frac{1}{4}\sin\left(\frac{x}{4}\right)$ and $\frac{dv}{dx} = -2\left(\sin\left(\frac{x}{4}\right)\right)^{-3}\left(\frac{1}{4}\cos\left(\frac{x}{4}\right)\right)$

$\rightarrow f''(x) = -\frac{1}{4}\cos\left(\frac{x}{4}\right) \times -2\left(\sin\left(\frac{x}{4}\right)\right)^{-3}\left(\frac{1}{4}\cos\left(\frac{x}{4}\right)\right) + \frac{1}{4}\sin\left(\frac{x}{4}\right)\left(\sin\left(\frac{x}{4}\right)\right)^{-2}$

$= \frac{1}{8}\cos^2\left(\frac{x}{4}\right)\left(\sin\left(\frac{x}{4}\right)\right)^{-3} + \frac{1}{16}\left(\sin\left(\frac{x}{4}\right)\right)^{-1}$

$= \frac{2\cos^2\left(\frac{x}{4}\right) + \sin^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)} = \boxed{\frac{1+\cos^2\left(\frac{x}{4}\right)}{16\sin^3\left(\frac{x}{4}\right)}}$ (Pythagorean identity $\sin^2\left(\frac{x}{4}\right) + \cos^2\left(\frac{x}{4}\right) = 1$)

- e.g. 2. If $y = e^{\sin^{-1}(x)}$, show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$

$\rightarrow \frac{dy}{dx} = \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}$ (since $\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$)

Let $u = e^{\sin^{-1}(x)}$ and $v = (1-x^2)^{1/2}$

$\frac{du}{dx} = \frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}$ $\qquad \frac{dv}{dx} = \frac{1}{2}(1-x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1-x^2}}$

$\rightarrow \frac{d^2y}{dx^2} = \frac{v \times \frac{du}{dx} - u \times \frac{dv}{dx}}{v^2} = \frac{e^{\sin^{-1}(x)} + \frac{xe^{\sin^{-1}(x)}}{\sqrt{1-x^2}}}{1-x^2}$

$= \boxed{\frac{\left(\sqrt{1-x^2} + x\right)e^{\sin^{-1}(x)}}{(1-x^2)^{3/2}}}$

$\rightarrow LHS = (1-x^2)\frac{\left(\sqrt{1-x^2}+x\right)e^{\sin^{-1}(x)}}{(1-x^2)^{3/2}} - x\left(\frac{e^{\sin^{-1}(x)}}{\sqrt{1-x^2}}\right) - e^{\sin^{-1}(x)}$

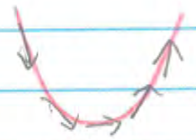
$= \frac{\sqrt{1-x^2}}{\sqrt{1-x^2}} + \frac{xe^{\sin^{-1}(x)}}{\sqrt{1-x^2}} - \frac{xe^{\sin^{-1}(x)}}{\sqrt{1-x^2}} - e^{\sin^{-1}(x)}$

$= \boxed{0}$ = RHS (as required)

Exercise 6E Points of inflexion

→ Concavity and points of inflection

- Concave up → if $f''(x) > 0$ for all $x \in (a,b)$
  - gradient of curve increasing
  - tangent below curve at each point



local min

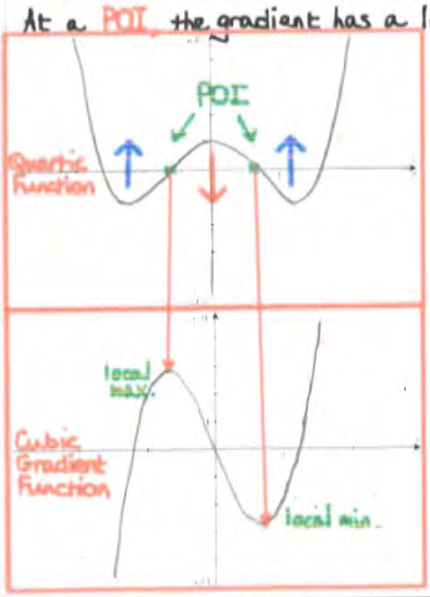
- Concave down → if $f''(x) < 0$ for all $x \in (a,b)$
  - gradient of curve decreasing
  - tangent above curve at each point

local max

- Point of inflexion (POI) → $\frac{d^2y}{dx^2} = 0$
  - crosses over from +ve to −ve or vice versa
  - instantaneous rate of change of gradient at pt. x
  - changes from concave up → concave down (vice versa)
  - sign of second derivative changes

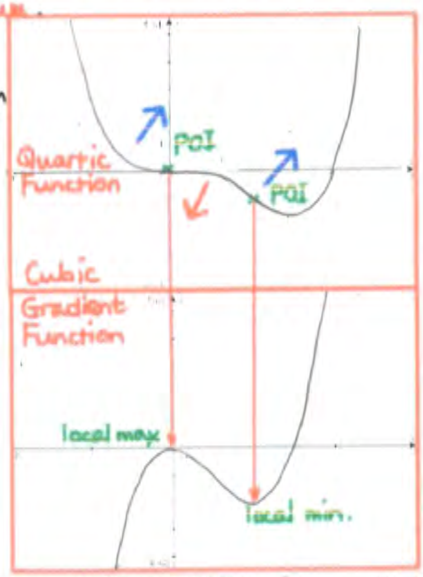
Relationship between POI and GRADIENT of a Function.

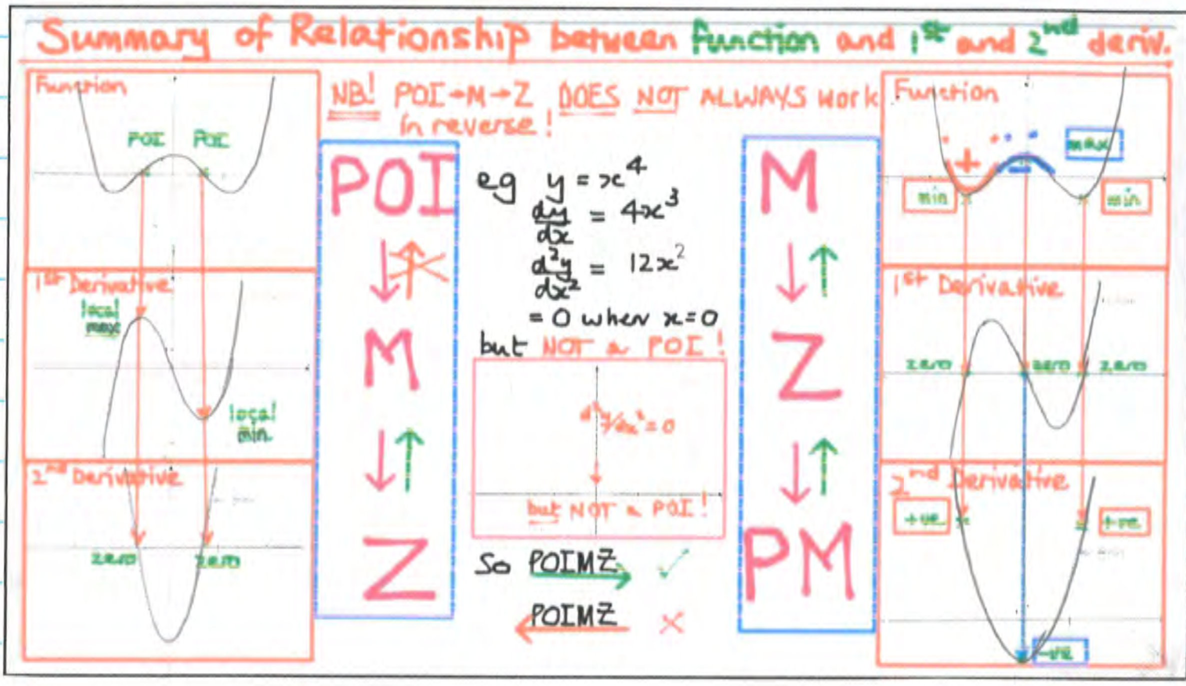
At a POI, the gradient has a local maximum or minimum.



The gradient trends toward a point of maximum steepness at the POI and then reverses the trend past the POI as the gradient trends in the opposite direction.

This accounts for the local max/min gradient at the POI.





→ Table of shapes - Gradients and Second Derivs.

2nd derivative

1st derivative

	$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$	$\frac{d^2y}{dy^2} = 0$
$\frac{dy}{dx} > 0$	Curve rising + concave up	Curve rising + concave down	P.O.I on rising curve
$\frac{dy}{dx} < 0$	Curve falling + concave up	Curve falling + concave down	P.O.I on falling curve
$\frac{dy}{dx} = 0$ (from Maths Methods)	Local min	Local max	Stationary P.O.I

Warning: $f''(x) = 0$

→ Always test nature of solns to $f''(x) = 0$

· Only solns corresponding to TURNING POINTS of $f'(x)$ are inflexion pts of $f(x)$

· Solns corresponding to stationary points of inflexion of $f'(x)$ are NOT inflexion pts of $f(x)$

eg $\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$, $\frac{dy}{dx} = 0$ when $x = 1$; $y = 0$ when $x = 3$.

(a) Find the x-co-ords of 2 POI's and 2 stationary points, showing that one POI is stationary.
(b) Find whether the stationary pt which is not a POI is a local maximum or minimum.
(c) Find y in terms of x.
(d) Find the co-ordinates of the stationary points, POI's and axes intercepts.
(e) Sketch y along with its first and second derivative functions on the same set of axes

(a) $\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$

Now at POI, $12x^2 - 36x + 24 = 0$
$\rightarrow x^2 - 3x + 2 = 0$
$\rightarrow (x-2)(x-1) = 0$
Hence $\boxed{x = 1 \text{ or } x = 2}$ (Inflexion pts)

Now testing the sign of the second derivative either side of these pts,
When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = +9$
When $x = \frac{3}{2}$, $\frac{d^2y}{dx^2} = -3$
When $x = \frac{5}{2}$, $\frac{d^2y}{dx^2} = +9$

$\frac{d^2y}{dx^2}$ + − 0 1 2 3

Hence, $\frac{d^2y}{dx^2}$ changes sign as required on either side of the zero pts., and hence $x = 1, 2$ are POI's

Now $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x + c$
Sub $\frac{dy}{dx} = 0$, $x = 1$ (given)

$\rightarrow 0 = 4(1)^3 - 18(1)^2 + 24(1) + c$
$\rightarrow 0 = 4 - 18 + 24 + c$
$\rightarrow 0 = 10 + c$
$\rightarrow c = -10$

Hence $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 10$
Now if $x = 1$, $\frac{dy}{dx} = 4 - 18 + 24 - 10 = 0$
Hence, $x = 1$ IS a stationary point and as it has been shown (opposite) to be a POI, it IS ALSO a stationary POI as req^d. (part (a))

Now $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 10$
$= (x-1)(4x^2 + \alpha x + 10)$
Expanding RHS "x^2" term,
$= -4x^2 + \alpha x^2 = (\alpha - 4)x^2$
Equating coefficients, $-18 = \alpha - 4$
$\rightarrow \alpha = -14$
Hence, $\frac{dy}{dx} = 2(x-1)(2x^2 - 7x + 5)$
$= 2(x-1)(x-1)(2x-5)$
Hence, stationary points occur at
$\boxed{x = 1 \text{ and } x = \frac{5}{2}}$ (Stationary pts)

From earlier, POI's occur at $x = 1$ and $x = 2$, so $x = 2$ is NOT a stationary POI (as required).

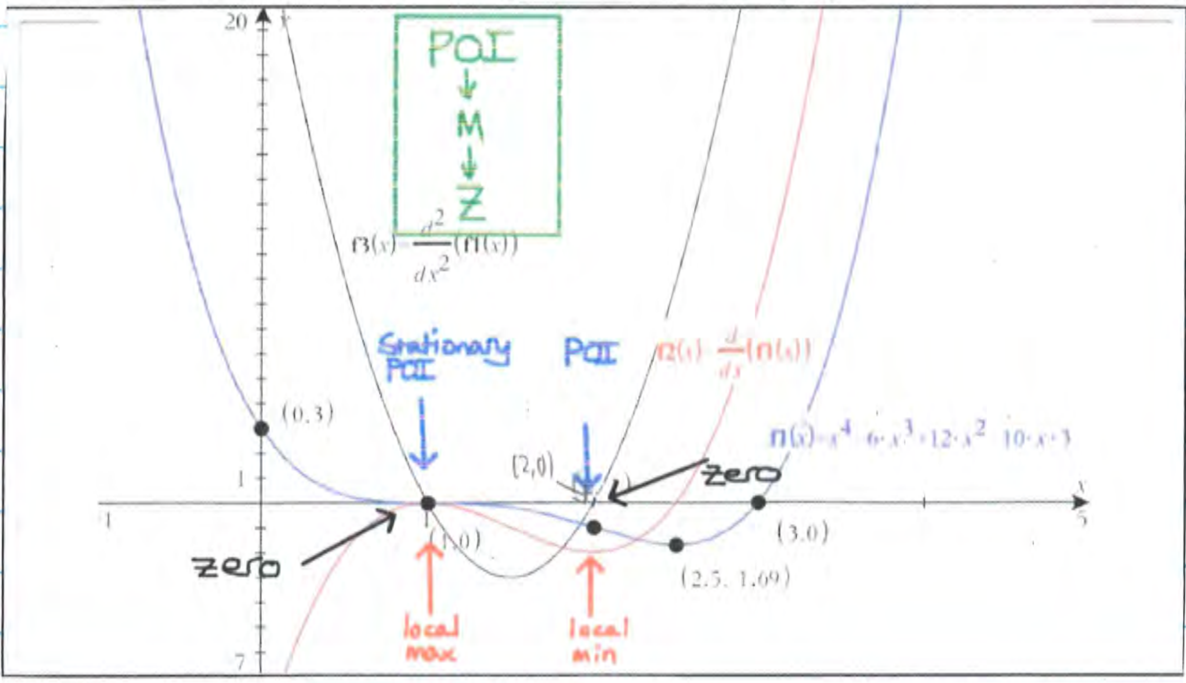
(b) Now $\frac{d^2y}{dx^2} = 12x^2 - 36x + 24$
When $x = 1$, $\frac{d^2y}{dx^2} = 0$, meaning that $x = 1$ is a stationary POI.
When $x = \frac{5}{2}$,
$\frac{d^2y}{dx^2} = 12\left(\frac{5}{2}\right)^2 - 36\left(\frac{5}{2}\right) + 24$
$= \frac{\cancel{12}^{3} \times 25}{\cancel{4}} - 90 + 24$
$= 75 - 90 + 24$
$= 99 - 90 = +9$

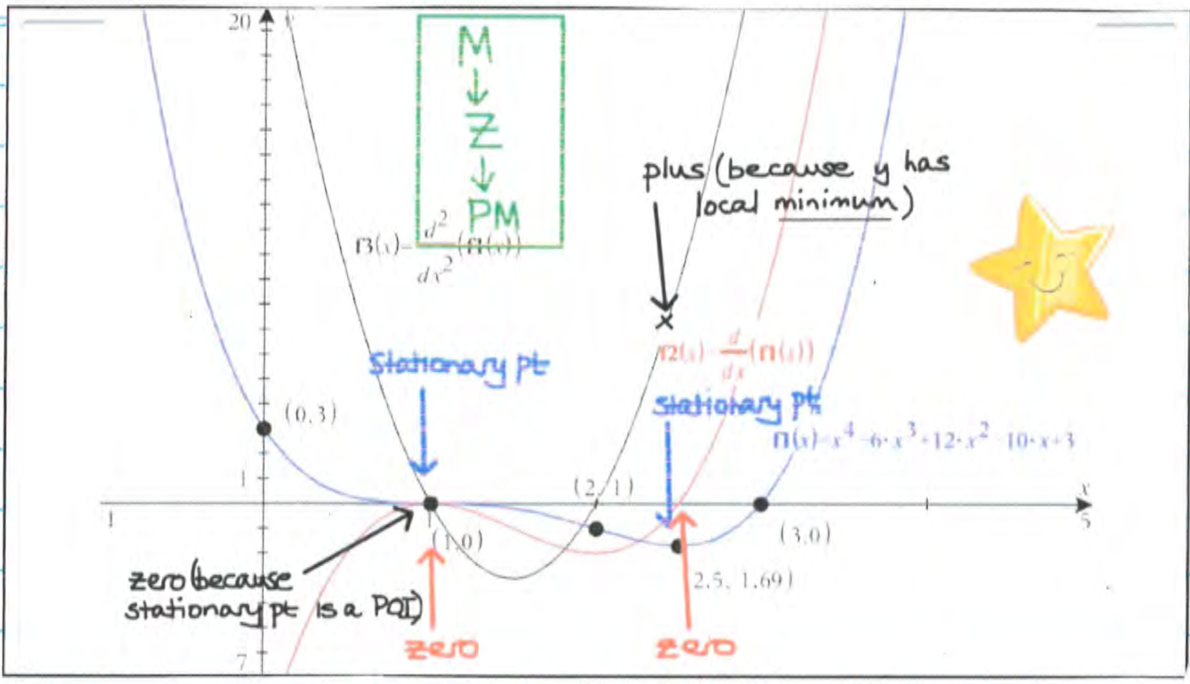
Hence because the second derivative is +ve, the stationary point at $x = \frac{5}{2}$ is a minimum.

(c) Now $\frac{dy}{dx} = 4x^3 - 18x^2 + 24x - 10$
$\rightarrow y = x^4 - 6x^3 + 12x^2 - 10x + d$

Sub $y = 0$, $x = 3$:-
$\rightarrow 0 = 3^4 - 6(3)^3 + 12(3)^2 - 10(3) + d$
$\rightarrow 0 = 81 - 6 \times 27 + 12 \times 9 - 30 + d$
$\rightarrow 0 = 81 - 162 + 108 - 30 + d$
$\rightarrow 0 = 189 - 192 + d$
$\rightarrow d = 3$
Hence, $\boxed{y = x^4 - 6x^3 + 12x^2 - 10x + 3}$

(d) When $x = 1$, $y = 0 \rightarrow (1, 0)$
Stationary POI and x-intercept.
When $x = 2$, $y = -1 \rightarrow (2, -1)$
Non-stationary POI
When $x = \frac{5}{2}$, $y = -\frac{27}{16} \rightarrow \left(\frac{5}{2}, -\frac{27}{16}\right)$
Stationary point - local minimum.
When $x = 3$, $y = 0 \rightarrow (3, 0)$ x-intercept
When $x = 0$, $y = +3 \rightarrow (0, 3)$ y-intercept





· e.g. 2 Find coordinates of all points of inflexion of $f(x) = \frac{3}{20}x^5 - x^4 + 2x^3 - 2$

$\rightarrow f'(x) = \frac{3}{4}x^4 - 4x^3 + 6x^2$

$\rightarrow f''(x) = 3x^3 - 12x^2 + 12x$

Solve $f''(x) = 3x^3 - 12x^2 + 12x = 0$,

$3x(x^2 - 4x + 4) = 0 \rightarrow 3x(x-2)^2 = 0 \rightarrow x = 0$ or $x = 2$

x	-1	0	1	2	3
$f''(x)$	-27	0	3	0	9
Shape	╲	—	╱	—	╱

$\rightarrow$ At $x=0$, $f'(x)$ has a minimum turning point.

Therefore, $f(x)$ has a point of inflexion at $x=0 \rightarrow y = f(0) = -2$

$\rightarrow$ At $x=2$, $f'(x)$ has a stationary point of inflexion, NOT a turning point. Therefore, $f(x)$ does NOT have a point of inflexion at $x=2$.

$\rightarrow$ Point of inflexion $\boxed{(0, -2)}$

· e.g. 3. Find the stationary points and the points of inflexion of the graph of $y = f(x) = 2x^2 \log_e(x)$.

$\rightarrow f'(x) = 4x\log_e(x) + 2x^2 \times \frac{1}{x}$

$= 4x\log_e(x) + 2x = 2x(1 + 2\log_e(x))$

$\rightarrow f''(x) = 2(1 + 2\log_e(x)) + 2x \times \frac{2}{x}$

$= 2(1 + 2\log_e(x)) + 4 = 2(3 + 2\log_e(x))$

$\rightarrow f'(x) = 0 \rightarrow 2x(1+2\log_e(x)) = 0$

$x = 0$ (reject) or $x = e^{-1/2}$ (accept) (since $x > 0$)

Stationary point: $\boxed{\left(\frac{1}{e^{1/2}}, -\frac{1}{e}\right)}$

$\rightarrow f''(x) = 0 \rightarrow 3 + 2\log_e(x) = 0 \rightarrow \log_e(x) = -\frac{3}{2} \rightarrow x = e^{-3/2}$

x	$\frac{1}{5}$	$e^{-3/2}$	$\frac{1}{2}$
$f''(x)$	-0.44	0	3.23
Shape	\	—	/

When $x = e^{-3/2}$, $f(x) = -3e^{-3}$

Point of inflexion $\boxed{(e^{-3/2}, -3e^{-3})}$

Exercise 6F Related rates

$\rightarrow$ Related rates steps (application of chain rule to rates of change)

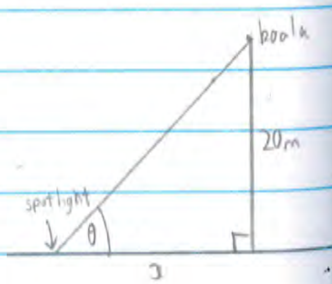
① Form equation (always on dt) $\rightarrow \frac{d(\,)}{dt} = \frac{d(\,)}{} \times \frac{}{dt}$

② Derive the unknown (use knowledge of area/volume formulas)

③ Differentiate and solve as necessary

2D shapes	3D shapes
Circle: $A = \pi r^2$, $P = 2\pi r$	Sphere: $TSA = 4\pi r^2$, $V = \frac{4}{3}\pi r^3$
Rectangle: $A = lw$, $P = 2l + 2w$	Cylinder: $TSA = 2\pi r^2 + 2\pi rh$, $V = \pi r^2 h$
Triangle: $A = \frac{1}{2}bh$, $P = b + h + \sqrt{b^2 + h^2}$,	Cone: $TSA = \pi r^2 + \pi rs$, $s = \sqrt{h^2 + r^2}$
$A = \frac{1}{2}bc\sin(A)$ (Pythagoras Theorem)	$V = \frac{1}{3}\pi r^2 h$

• e.g. 1. A park ranger stands at the base of a tree and shines a spotlight on a stationary koala 20m vertically above the spotlight. The angle made between the horizontal ground and the light beam to the koala is θ. The ranger moves in a straight line away from the base of the tree at a speed of $1.5\ ms^{-1}$, while still shining the spotlight on the stationary koala. Find the rate at which θ is decreasing when $\theta = \frac{\pi}{3}$.



→ Let x be distance between spotlight and base of tree at time t.

- Given rate: $\frac{dx}{dt} = \frac{3}{2}\ ms^{-1}$ - Required rate: $\frac{d\theta}{dt}$ when $\theta = \frac{\pi}{3}$
- Chain rule: $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} \rightarrow \frac{d\theta}{dt} = \frac{3}{2} \times \frac{d\theta}{dx}$ (x must be defined!)

→ $\tan(\theta) = \frac{20}{x} \rightarrow \theta = \tan^{-1}\left(\frac{20}{x}\right)$

$\frac{d\theta}{dx} = \frac{1}{1+\left(\frac{20}{x}\right)^2} \times \left(-\frac{20}{x^2}\right)$

→ $\theta = \frac{\pi}{3} \rightarrow \tan\left(\frac{\pi}{3}\right) = \frac{20}{x} \rightarrow \frac{20}{x} = \sqrt{3} \rightarrow x = \frac{20\sqrt{3}}{3}$

When $\theta = \frac{\pi}{3}$, $\frac{d\theta}{dx} = \frac{1}{1+(\sqrt{3})^2} \times \left(\frac{-20}{400/3}\right) \rightarrow \frac{d\theta}{dx} = -\frac{3}{80}$

→ Sub $\frac{d\theta}{dx} = -\frac{3}{80}$, $\frac{d\theta}{dt} = \frac{3}{2} \times \left(-\frac{3}{80}\right) = -\frac{9}{160}$

Rate of decrease: $\boxed{\frac{9}{160} \text{ radians per second}}$

- e.g. 2. Chemicals are added to a container so that a particular crystal will grow in the shape of a cube. The side length of the crystal, x cm, t days after the chemicals were added to the container, is given by $x = \arctan(2t)$. Find the rate at which the surface area of the crystal is growing one day after the chemicals were added. Give your answer in square millimetres per day. (VCAA 2016 Exam 1 Q4)

→ Surface area: $A = 6x^2\ cm^2 \rightarrow A = 600x^2\ mm^2$ (since $1 cm^2 = 100 mm^2$)

$\therefore A = 600x^2$ where $x = \arctan(2t)$

→ $\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = 1200x \times \left(\frac{1/2}{\left(\frac{1}{2}\right)^2 + t^2}\right)$

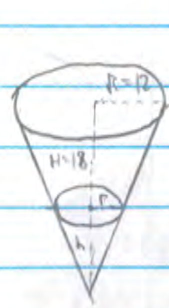
$= 1200x\left(\frac{2}{1+4t^2}\right) = \frac{2400x}{1+4t^2}$

→ Sub $t = 1$ into $x = \arctan(2t)$: $x = \arctan(2)$

Sub $t=1$ and $x = \arctan(2)$ into $\frac{dA}{dt} = \frac{2400x}{1+4t^2}$:

$\frac{dA}{dt} = \frac{2400\arctan(2)}{1+4(1)^2} = \boxed{480 \arctan(2)\ mm^2/day}$

- e.g. 3. A cone of radius (R) 12 cm and height (H) 18 cm is filled with water at a rate of $2 cm^3/s$. Find the rate of increase of the height of the water where the depth of the water is 6 cm.



→ $\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt} \rightarrow \frac{dh}{dt} = 2 \times \frac{dV}{dt}$

→ $V = \frac{1}{3}\pi r^2 h$, from diagram $\frac{r}{h} = \frac{12}{18}$ (similar triangles)

$\therefore r = \frac{2h}{3} \rightarrow V = \frac{4}{27}\pi h^3$

→ $\frac{dV}{dh} = \frac{12}{27}\pi h^2 \rightarrow \frac{dh}{dV} = \frac{27}{12\pi h^2}$

→ $\frac{dh}{dt} = 2 \times \frac{27}{12\pi(6)^2} = 2 \times \frac{1}{16\pi} = \frac{1}{8\pi}$ cm/s

$\therefore$ Rate of increase: $\boxed{\frac{1}{8\pi} \text{ cm/s}}$

→ Parametric equations

- Parametric curve → pair of functions
  - $x = f(t)$ and $y = g(t)$
  - t (parameter) → point in plane $(f(t), g(t))$
- Gradient at a point on a parametric curve

$$\rightarrow \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \text{ provided } \frac{dx}{dt} \neq 0$$

- Second derivative at a point on a parametric curve

$$\rightarrow \frac{d^2y}{dx^2} = \frac{d(y')}{dx} = \frac{dy'}{dt} \times \frac{dt}{dx}, \text{ where } y' = \frac{dy}{dx}$$

- CAS calculator (Graph application)
  → Menu → Graph Entry/Edit (3) → Parametric (4)
- e.g. 1. A curve is defined by parametric equations $x = \sec(t)$ and $y = \tan(t)$.
  (a) Find the equation of the normal to the curve at point $(\sec(t), \tan(t))$
  (b) Let A and B be points of intersection of normal to curve with x-axis and y-axis respectively, and let O be the origin. Find the area of $\triangle OAB$.
  (c) Find the value of t for which area of $\triangle OAB$ is $4\sqrt{3}$.

(a) $\frac{dy}{dt} = \sec^2(t)$, $\frac{dx}{dt} = \sec(t)\tan(t)$

$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\sec^2(t)}{\sec(t)\tan(t)} = \sec(t) \times \frac{\cos(t)}{\sin(t)} = \operatorname{cosec}(t)$

→ Gradient of normal $= -\frac{1}{\operatorname{cosec}(t)} = -\sin(t)$

→ Equation of normal: $y - \tan(t) = -\sin(t)(x - \sec(t))$

$\boxed{y = -\sin(t)x + 2\tan(t)}$

(b) When $x = 0$, $y = 2\tan(t)$

When $y = 0$, $x = \frac{2\tan(t)}{\sin(t)} = \frac{2}{\cos(t)} = 2\sec(t)$

→ $O = (0,0)$, $A = (2\sec(t), 0)$, $B = (0, 2\tan(t))$

Area $= \frac{1}{2}|4\sec(t)\tan(t)| = \left|2 \times \frac{1}{\cos(t)} \times \frac{\sin(t)}{\cos(t)}\right|$

$= \boxed{\frac{2\sin(t)}{\cos^2(t)}}$

(c) Assume $0 \leq t < \frac{\pi}{2}$

→ $\frac{2\sin(t)}{\cos^2(t)} = 4\sqrt{3}$ → $\sin(t) = 2\sqrt{3}(1 - \sin^2(t))$

$2\sqrt{3}\sin^2(t) + \sin(t) - 2\sqrt{3} = 0$

→ $\sin(t) = \frac{-1 \pm \sqrt{1 - 4(2\sqrt{3})(-2\sqrt{3})}}{4\sqrt{3}} = \frac{-1 \pm 7}{4\sqrt{3}}$

$\sin(t) = -\frac{2\sqrt{3}}{3}$ (reject) or $\sin(t) = \frac{\sqrt{3}}{2}$ (accept)

→ $\boxed{t = \frac{\pi}{3}}$

· e.g. 2. For the parametric curve given by $x = t^2 + 1$ and $y = t(t-3)^2$, for $t \in \mathbb{R}$, find $\frac{d^2y}{dx^2}$ and coordinates of points of inflexion.

$\rightarrow \frac{dy}{dt} = 3t^2 - 12t + 9$, $\frac{dx}{dt} = 2t$ (since $y = t^3 - 6t^2 + 9t$)

$\rightarrow \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2 - 12t + 9}{2t} = \frac{3(t-1)(t-3)}{2t}$, $t \neq 0$

$\rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{3}{2}\left(t - 4 + \frac{3}{t}\right)\right) = \frac{dy'}{dt} \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{3}{2}\left(t - 4 + \frac{3}{t}\right)\right) \times \frac{1}{2t}$

$= \left(\frac{3}{2} - \frac{9}{2t^2}\right) \times \frac{1}{2t} = \boxed{\frac{3(t^2-3)}{4t^3}}$

$\rightarrow \frac{d^2y}{dx^2} = 0 \rightarrow t = \pm\sqrt{3}$

When $t = \sqrt{3}$, $x = 4$ and $y = 12\sqrt{3} - 18$.

When $t = -\sqrt{3}$, $x = 4$ and $y = -12\sqrt{3} - 18$

∴ Points of inflexion: $\boxed{(4, 12\sqrt{3} - 18), (4, -12\sqrt{3} - 18)}$

Exercise 6G Graphs of rational functions

$\rightarrow$ Graphing rational functions $\left(f(x) = \frac{P(x)}{Q(x)}\right)$

· divide denominator into numerator when degree of denominator is not less than degree of numerator

$\rightarrow$ able to find trends, asymptotes

· Features: asymptotes, axis intercepts, stationary points, points of inflexion

· Methods: add y-coordinates (ordinates) of two simple graphs; take reciprocals of y-coordinates (ordinates) of a simple graph

* · N.B. Draw single-line graphs, NOT multi-line 'feathered' style.

$\rightarrow$ Addition of ordinates

· when two graphs have same ordinate $\rightarrow$ y-coordinate of resultant graph double

· when two graphs have opposite ordinates $\rightarrow$ y-coordinate zero (x-intercept)

· when one of two ordinates is zero $\rightarrow$ resulting ordinate equal to other ordinate

$\rightarrow$ Key features: $\left(f(x) = \frac{a(x)}{b(x)} = q(x) + \frac{r(x)}{b(x)}\right)$

· Vertical asymptotes $\rightarrow b(x) = 0$

· Non-vertical asymptote: $y = q(x)$

} N.B. Graphs must approach asymptote as appropriate, NOT swing away after approaching!

· x-intercepts: $a(x) = 0$

· y-intercept: $f(0) = \frac{a(0)}{b(0)}$ (if $b(0) \neq 0$)

→ Reciprocal of ordinates

Function $y = f(x)$	Reciprocal function $y = \frac{1}{f(x)}$
x-axis intercept at $x = a$	vertical asymptotes $x = a$
local max at $x = a$	local min at $x = a$
local min at $x = a$	local max at $x = a$
above x-axis	above x-axis
below x-axis	below x-axis
increasing over an interval	decreasing over an interval
decreasing over an interval	increasing over an interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$
pass through $f(x) = \pm 1$	pass through $f(x) = \pm 1$.

→ Graph examples

e.g. 1. Sketch graph of $f(x) = \frac{4+x^2+x^3}{x}$ for $x \in [-3,3]$, labelling the turning point and point of inflexion with their coordinates, correct to two decimal places. (VCAA 2016 Exam 2 Q1)

→ Stationary point, $f'(x) = 0 \rightarrow x = 1.113386$..

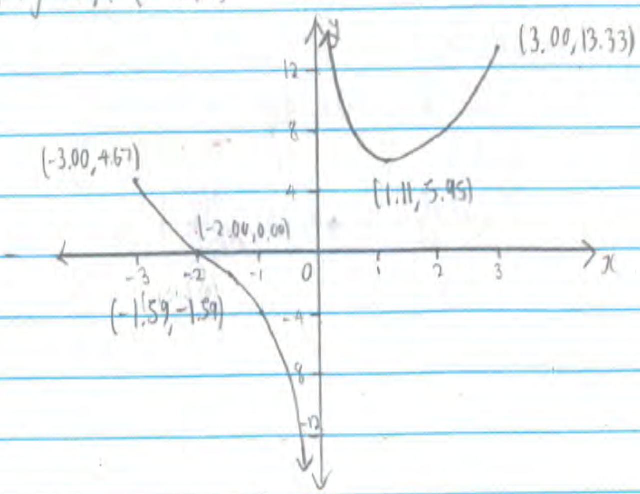
$f(1.113386) = 5.94566 \rightarrow (1.11, 5.95)$

→ POI: $f''(x) = 0 \rightarrow x = -1.5874$.

$f(-1.5874) = -1.5874 \rightarrow (-1.59, -1.59)$

→ $f(x) = \frac{4}{x} + x + x^2$ → Asymptotes: $x = 0$, $y = x^2 + x$, $y = \frac{4}{x}$

→ x-intercept $(y = 0)$: $(-2, 0)$



e.g. 2. Sketch the graph of $g: [-5,5] \to R$, where $g(x) = \frac{4x^2}{x^2+2}$.
(Include stationary points and points of inflexion)

→ Stationary point: $g'(x) = \frac{16x}{(x^2+2)^2} = 0 \to x = 0$

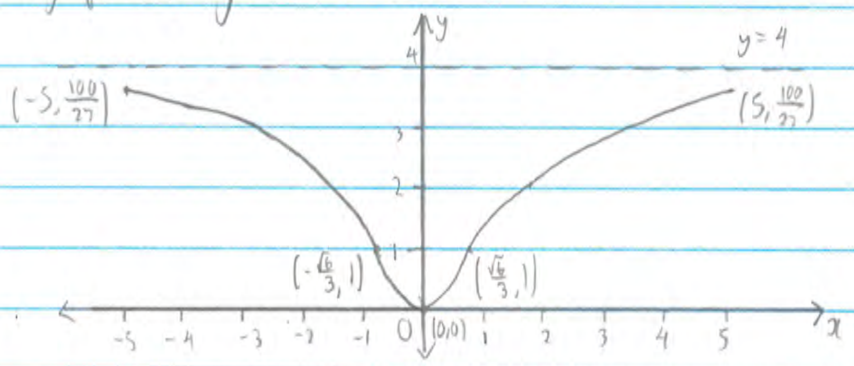
$g(0) = 0 \to (0,0)$

→ POI: $g''(x) = \frac{16(2-3x^2)}{(x^2+2)^3} = 0 \to x = \pm\frac{\sqrt{6}}{3}$

$\therefore \left(-\frac{\sqrt{6}}{3}, 1\right), \left(\frac{\sqrt{6}}{3}, 1\right)$

→ $g(x) = \frac{4(x^2+2)-8}{x^2+2} = 4 - \frac{8}{x^2+2}$

Asymptotes: $y = 4$



e.g. 3. Sketch the graph of $y = \frac{2}{x^2} - \frac{x}{2}$. Give exact coordinates of any turning points and intercepts, and state the equations of all straight line asymptotes. (VCAA 2008 Exam 1 Q1)

→ Asymptotes: $x = 0,\ y = -\frac{x}{2}$

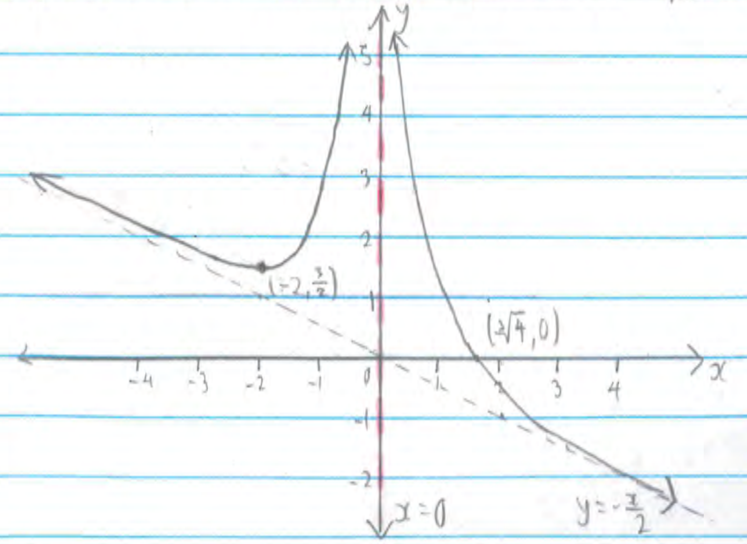
→ x-intercept ($y=0$): $0 = \frac{2}{x^2} - \frac{x}{2} \to 0 = 2 - \frac{x^3}{2}$

$x = \sqrt[3]{4} \to (\sqrt[3]{4}, 0)$

→ Turning point: $\frac{dy}{dx} = -\frac{4}{x^3} - \frac{1}{2} = 0 \to x^3 = -8 \to x = -2$

When $x = -2$, $y = \frac{2}{4} - \frac{(-2)}{2} = \frac{3}{2} \to \left(-2, \frac{3}{2}\right)$

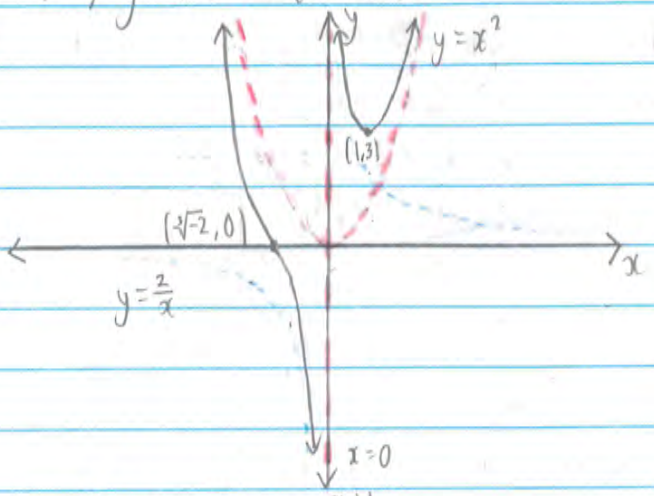
→ POI: $\frac{d^2y}{dx^2} = \frac{12}{x^4} = 0 \to$ No solns → No inflexion points



- e.g. 4. Sketch the graph of $y = \frac{x^3+2}{x}$. $\left(= x^2 + \frac{2}{x}\right)$
  - → Asymptotes: $x = 0$, $y = x^2$, $y = \frac{2}{x}$
  - → Turning points: $\frac{dy}{dx} = 2x - \frac{2}{x^2} = 0 \rightarrow 2x^3 - 2 = 0 \rightarrow x = 1$

    When $x = 1$, $y = 3$ → $(1,3)$
  - → POI: $\frac{d^2y^2}{dx^2} = 2 + \frac{4}{x^3} = 0 \rightarrow x = \sqrt[3]{-2}$

    When $x = \sqrt[3]{-2}$, $y = 0$. → $(\sqrt[3]{-2}, 0)$

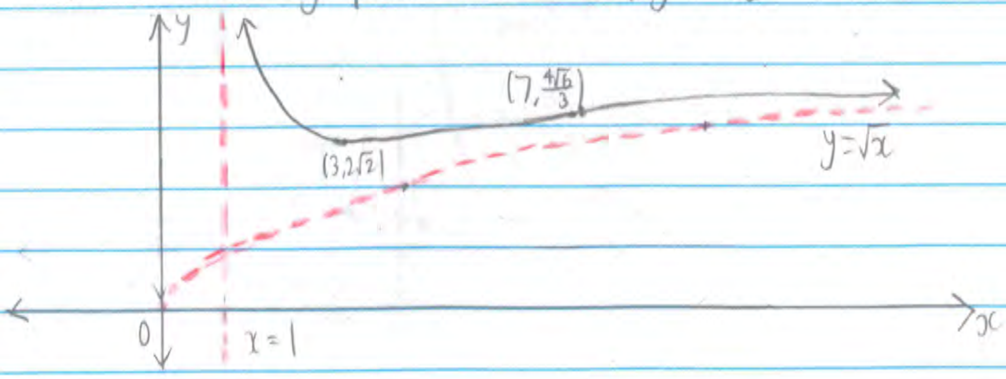


- e.g. 5. Sketch the graph of $y = \frac{x+1}{\sqrt{x-1}}$.
  - → Maximal domain: $x - 1 > 0 \rightarrow x > 1$
  - → Let $u = x + 1$ and $v = (x-1)^{1/2}$

    $\frac{du}{dx} = 1$ $\quad\quad$ $\frac{dv}{dx} = \frac{1}{2(x-1)^{1/2}}$

    $$\frac{dy}{dx} = \frac{1 \times \sqrt{x-1} - \frac{x+1}{2(x-1)^{1/2}}}{x-1} = \frac{x-3}{2(x-1)^{3/2}}$$
  - → $\frac{dy}{dx} = 0 \rightarrow x = 3$

    When $x = 3$, $y = \frac{4}{\sqrt{2}} = 2\sqrt{2}$ → $(3, 2\sqrt{2})$ (local min)
  - → Solve $\frac{d^2y}{dx^2} = 0$, $x = 7$ → When $x = 7$, $y = \frac{4\sqrt{6}}{3}$ → $\left(7, \frac{4\sqrt{6}}{3}\right)$ (POI)
  - → Vertical asymptote: $x = 1$
  - → Non-vertical asymptote: As $x \rightarrow \infty$, $y \rightarrow \sqrt{\frac{x^2}{x}} = \sqrt{x}$



Exercise 6I Implicit differentiation

→ Implicit differentiation → differentiating 'term by term' as you go WITHOUT re-arranging - use of chain + product rules apply

- used when we CAN'T make 'y' subject to enable use of 'traditional' method → for complex rules, not of form $y=f(x)$ or $x=f(y)$

- e.g. 1. Find the gradient of the normal to the curve defined by $y=-3e^{3x}e^{y}$ at the point $(1,-3)$ (VCAA 2014 Exam 1 Q4)

$\rightarrow \frac{dy}{dx} = -3\left[e^{3x}\times e^{y}\frac{dy}{dx} + e^{y}\times 3e^{3x}\right]$

$= -3e^{3x}e^{y}\frac{dy}{dx} - 9e^{3x}e^{y}$

$\frac{dy}{dx}\left[1+3e^{3x}e^{y}\right] = -9e^{3x}e^{y}$

$\rightarrow \frac{dy}{dx} = \frac{-9e^{3x}e^{y}}{1+3e^{3x}e^{y}}$

Sub $x=1, y=-3$: $\frac{dy}{dx} = \frac{-9e^{3}e^{-3}}{1+3e^{3}e^{-3}} = -\frac{9}{4}$

$\therefore$ Gradient of normal $= \boxed{+\frac{4}{9}}$

- e.g. 2. Consider the curve represented by $x^2 - xy + \frac{3}{2}y^2 = 9$. (VCAA 2015 Exam 1 Q9)

(a) Find the gradient of the curve at any point (x,y).

(b) Find the equation of the tangent to the curve at the point $(3,0)$ and find the equation of the tangent at the point $(0,\sqrt{6})$. Write each equation in the form $y=ax+b$.

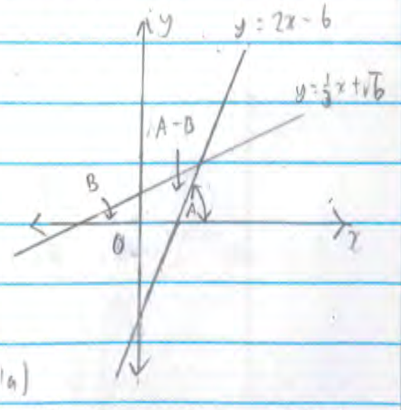
(c) Find the acute angle between the tangent to the curve at the point $(0,\sqrt{6})$. Give your answer in the form $k\pi$, where $k\in R$.

(a) $2x - \left(x\times\frac{dy}{dx} + 1\times y\right) + \frac{d}{dy}\left(\frac{3}{2}y^2\right)\times\frac{dy}{dx} = 0 \rightarrow 2x - x\left(\frac{dy}{dx}\right) - y + 3y\left(\frac{dy}{dx}\right) = 0$

$2x - y + \frac{dy}{dx}\left[-x+3y\right] = 0$ (use of product and chain rules)

$\frac{dy}{dx}\left[3y-x\right] = y-2x$

$\rightarrow \frac{dy}{dx} = \boxed{\frac{y-2x}{3y-x}}$



(b) Sub $x=3, y=0$: $\frac{dy}{dx} = \frac{0-6}{0-3} = 2$

$y-0 = 2(x-3) \rightarrow \boxed{y=2x-6}$

Sub $x=0, y=\sqrt{6}$: $\frac{dy}{dx} = \frac{\sqrt{6}-0}{3\sqrt{6}-0} = \frac{1}{3}$

$y-\sqrt{6} = \frac{1}{3}(x-0) \rightarrow \boxed{y=\frac{1}{3}x+\sqrt{6}}$

(c) • $\tan(A-B) = \frac{\tan(A)-\tan(B)}{1+\tan(A)\tan(B)} = \frac{2-\frac{1}{3}}{1+2\left(\frac{1}{3}\right)} = \frac{5}{3}\div\frac{5}{3} = 1$

$A-B = \tan^{-1}(1) = \boxed{\frac{\pi}{4}}$ ('compound' angle formula)

• Let $y=2x-6$ be vector $\underset{\sim}{a} = \underset{\sim}{i}+2\underset{\sim}{j}$, and $y=\frac{1}{3}x+\sqrt{6}$ be vector $\underset{\sim}{b} = 3\underset{\sim}{i}+\underset{\sim}{j}$

$\cos(\theta) = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}|\times|\underset{\sim}{b}|} = \frac{3+2}{\sqrt{5}\times\sqrt{10}} = \frac{5}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \rightarrow \boxed{\theta = \frac{\pi}{4}}$ (vector method)

e.g.3. Find the value of c, where $c \in R$, such that the curve defined by: $y^2 + \frac{3e^{(x-1)}}{x-2} = c$, has a gradient of 2 when $x = 1$.
(VCAA 2013 Exam 1 Q6)

When $x = 1$, $y^2 + \frac{3e^0}{1-2} = c \rightarrow y^2 - 3 = c \cdots ①$

$$\rightarrow 2y\left(\frac{dy}{dx}\right) + \frac{(x-2) \times 3e^{(x-1)} - 3e^{(x-1)}}{(x-2)^2} = 0$$

$$\frac{3e^{(x-1)}(x-3)}{(x-2)^2} = -2y\left(\frac{dy}{dx}\right) \rightarrow \frac{dy}{dx} = \frac{-3(x-3)e^{(x-1)}}{2y(x-2)^2}$$

When $x = 1$, $\frac{dy}{dx} = \frac{-3(-2)e^0}{2y} = \frac{3}{y} = 2 \rightarrow y = \frac{3}{2}$

Sub $y = \frac{3}{2}$ into ①, $c = \left(\frac{3}{2}\right)^2 - 3 \rightarrow \boxed{c = -\frac{3}{4}}$

e.g.4. Find the gradient of the line perpendicular to graph of $\log_e(2xy) + \frac{x}{y} = 8$ at the point $(2, \frac{1}{4})$.

$\rightarrow \log_e(2) + \log_e(x) + \log_e(y) + xy^{-1} = 8$ (log laws)

$\frac{1}{x} + \frac{1}{y}\left(\frac{dy}{dx}\right) + \frac{1}{y} - \frac{x}{y^2}\left(\frac{dy}{dx}\right) = 0$

$\frac{1}{x} + \frac{1}{y} = \frac{dy}{dx}\left(\frac{x}{y^2} - \frac{1}{y}\right) \rightarrow \frac{x+y}{xy} = \frac{dy}{dx}\left(\frac{x-y}{y^2}\right)$

$\frac{dy}{dx} = \frac{y(x+y)}{x(x-y)}$

When $x = 2$ and $y = \frac{1}{4}$, $\frac{dy}{dx} = \frac{\frac{1}{4}(2+\frac{1}{4})}{2(2-\frac{1}{4})} = \frac{1/4 \times 9/4}{4 - 1/2}$

$= \frac{9/16}{7/2} = \frac{9}{16} \times \frac{2}{7} = \frac{9}{56}$

$\therefore$ Gradient of normal: $m_N = \boxed{-\frac{56}{9}}$

e.g.5. Find the equation of the tangent to the curve $\log_e\left(\frac{x}{y}\right) + x^2 - y = 0$ at the point $(1,1)$ in the form $ax + by + c = 0$ $(x, y > 0)$

$\rightarrow \log_e(x) - \log_e(y) + x^2 - y = 0$

$\frac{1}{x} - \frac{1}{y}\left(\frac{dy}{dx}\right) + 2x - \frac{dy}{dx} = 0$

$\frac{dy}{dx}\left(\frac{1}{y} + 1\right) = 2x + \frac{1}{x}$

When $x = 1$ and $y = 1$, $\frac{dy}{dx}(1+1) = 2+1 \rightarrow \frac{dy}{dx} = \frac{3}{2}$

$y - 1 = \frac{3}{2}(x-1) \rightarrow y = \frac{3}{2}x - \frac{1}{2}$

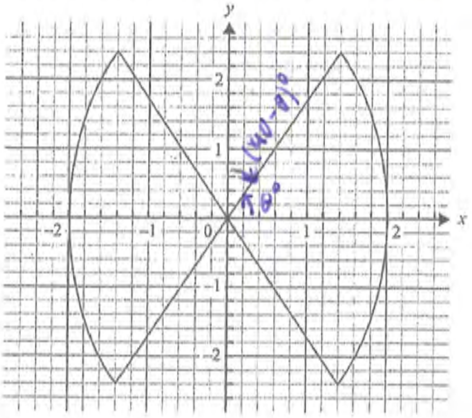
$2y = 3x - 1 \rightarrow \boxed{3x - 2y - 1 = 0}$

VCAA Application Questions – Miscellaneous (2017)

VCAA 2017 Exam 2 Q3

Question 3 (10 marks)

A brooch is designed using inverse circular functions to make the shape shown in the diagram below.



The edges of the brooch in the first quadrant are described by the piecewise function

$$f(x) = \begin{cases} 3\arcsin\left(\frac{x}{2}\right), & 0 \le x \le \sqrt{2} \\ 3\arccos\left(\frac{x}{2}\right), & \sqrt{2} < x \le 2 \end{cases}$$

a. Write down the coordinates of the corner point of the brooch in the first quadrant. (1 mark)

$f(\sqrt{2}) = 3\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

$\therefore \boxed{\left(\sqrt{2}, \frac{3\pi}{4}\right)}$

b. Specify the piecewise function that describes the edges in the third quadrant. (1 mark)

$$y(x) = \begin{cases} -3\cos^{-1}\left(-\frac{x}{2}\right), & -2 \le x < -\sqrt{2} \\ -3\sin^{-1}\left(+\frac{x}{2}\right), & -\sqrt{2} \le x \le 0 \end{cases}$$

Reflection in x and y-axis

c. Given that each unit in the diagram represents one centimetre, find the area of the brooch. Give your answer in square centimetres, correct to one decimal place. (3 marks)

Area = 4 × area of 1st quadrant

$= 4 \times \left[\int_0^{\sqrt{2}} 3\sin^{-1}\left(\frac{x}{2}\right) dx + \int_{\sqrt{2}}^{2} 3\cos^{-1}\left(\frac{x}{2}\right) dx\right]$

$= 9.9411\ldots \to \boxed{9.9 \text{ cm}^2}$ (1 dp)

d. Find the acute angle between the edges of the brooch at the origin. Give your answer in degrees, correct to one decimal place. (3 marks)

For $0 \le x \le \sqrt{2}$, $f'(x) = \frac{3}{\sqrt{4-x^2}}$

$f'(0) = \frac{3}{\sqrt{4}} = \frac{3}{2}$

$\theta = \tan^{-1}\left(\frac{3}{2}\right) = 56.3099°$

$\to$ Acute angle $= 2 \times (90° - \theta)$

$= 2 \times (90° - 56.3099°)$

$= 67.38° \to \boxed{67.4°}$ (1 dp)

e. The perimeter of the brooch has a border of gold.

Show that the length of the gold border needed is given by a definite integral of the form $\int_0^2 \left(\sqrt{a + \frac{b}{4-x^2}}\right) dx$, where $a, b \in R$. Find the values of a and b. (2 marks)

e Use arc length $= \int_a^b \sqrt{1 + (f'(x))^2}\, dx$

Total arc length $= 4 \times \left(\int_0^{\sqrt{2}} \sqrt{1 + \left(\frac{d}{dx}\left(3\sin^{-1}\left(\frac{x}{2}\right)\right)\right)^2}\, dx + \int_{\sqrt{2}}^{2} \sqrt{1 + \left(\frac{d}{dx}\left(3\cos^{-1}\left(\frac{x}{2}\right)\right)\right)^2}\, dx\right)$

$= 4\left(\int_0^{\sqrt{2}} \sqrt{1 + \left(\frac{3}{\sqrt{4-x^2}}\right)^2}\, dx + \int_{\sqrt{2}}^{2} \sqrt{1 + \left(\frac{-3}{\sqrt{4-x^2}}\right)^2}\, dx\right)$

$= 4\left(\int_0^2 \sqrt{1 + \left(\frac{9}{4-x^2}\right)}\, dx\right)$

$= \int_0^2 \sqrt{16 + \left(\frac{144}{4-x^2}\right)}\, dx$ where $a = 16$, $b = 144$

VCAA 2017 Exam 1 Q6

Question 6 (3 marks)

Let $f(x) = \frac{1}{\arcsin(x)}$.

Find $f'(x)$ and state the largest set of values of x for which $f'(x)$ is defined.

$f(x) = (\sin^{-1}(x))^{-1}$

$\to f'(x) = -(\sin^{-1}(x))^{-2} \times \frac{1}{\sqrt{1-x^2}}$

$= \frac{-1}{(\sin^{-1}(x))^2\sqrt{1-x^2}}$

Terms on the denominator can't be zero so for $\sqrt{1-x^2}$, $x \ne -1, 1$.

and for $\sin^{-1}(x)$, $x \ne 0$

so $x \in [-1, 0) \cup (0, 1]$ (instead of $[-1, 1]$)

The domain for which $f'(x)$ is defined is the intersection of that of the two denominator terms, so

$x \in (-1, 0) \cup (0, 1)$

$\to \boxed{f'(x) = \frac{-1}{(\arcsin(x))^2\sqrt{1-x^2}}}$

$\boxed{x \in (-1, 0) \cup (0, 1)}$

Question 8 VCAA 2017

Let $f(x) = x^3 - mx^2 - 4$, where $m, x \in R$.

The **gradient** of f will always be strictly increasing for

A. $x \geq 0$

B. $x \geq \frac{m}{3}$

C. $x \leq \frac{m}{3}$

D. $x \geq \frac{2m}{3}$

E. $x \leq \frac{2m}{3}$

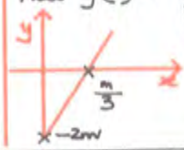
Let gradient of $f(x)$ be $g(x)$.

$g(x) = 3x^2 - 2mx$

Now $g(x)$ will be strictly increasing for sections of the domain where the gradient is ≥ 0 according to the rules

for s.i.

Now $g'(x) = 6x - 2m$ (2nd derivative!)



For this graph, the x-intercept (where $y=0$) is $\frac{m}{3}$

Hence $x \geq \frac{m}{3}$ for s.i.

→ B

Question 10 VCAA 2017

A function f, its derivative f' and its second derivative f'' are defined for $x \in R$ with the following properties.

$f(a) = 1, f(-a) = -1$

$f(b) = -1, f(-b) = 1$

and $f''(x) = \frac{(x+a)^2(x-b)}{g(x)}$, where $g(x) < 0$

The coordinates of any points of inflection of $|f(x)|$ are

A. $(-a, 1)$ and $(b, 1)$

B. $(b, -1)$

C. $(-a, -1)$ and $(b, -1)$

D. $(-a, 1)$

E. $(b, 1)$

If $f''(x) = \frac{(x+a)^2(x-b)}{g(x)}$

then $f''(x) = 0$ when $x = -a$ and when $x = b$.

However we need to verify that $f''(x)$ also changes sign from + to − or vice versa through zero where $f''(x) = 0$

Now $f''(x) = \frac{(x+a)^2(x-b)}{g(x)}$

$(x+a)^2$ is always +ve for any x.

$g(x)$ is always −ve for any x.

Hence, the only way to change the sign of $f''(x)$ depends on the term $(x-b)$

If $x > b$, $(x-b)$ is +ve

If $x < b$, $(x-b)$ is −ve

So $f''(x)$ changes sign through zero at $x = b$.

Hence there is a POI at $(b, -1)$ on $f(x)$.

So for $|f(x)|$, the POI would occur at $(b, 1)$ → E

The function f is defined by $f(x) = \dfrac{x^3 - a^3}{x^2 - 4}$ where $a > 2$. The graph of $y = f(x)$ has two vertical asymptotes and a non-vertical asymptote.

a. Find the equations of the three asymptotes. 3 marks

$f(x) = x + \dfrac{4x - a^3}{x^2 - 4}$ (Menu → 2 → 7 → 1)

Asymptotes: $\boxed{y = x,\ x = -2,\ x = 2}$

b. i. Show that the graph of f has a stationary point at $x = 0$. 2 marks

$f'(x) = \dfrac{x(x^3 - 12x + 2a^3)}{(x^2-4)^2}$

$f'(x) = 0$ when $x = 0$

→ Stationary point at $x = 0$ (Shown)

ii. Find $f''(0)$ and hence justify the nature of the stationary point at $x = 0$. 2 marks

$f''(0) = a^3/8$

$\dfrac{a^3}{8} > 0$ for $a > 2$

→ $x = 0$ is a local minimum

c. Show algebraically that the graph of $y = f(x)$ crosses the non-vertical asymptote. 2 marks

$x^3 - a^3 = x(x^2 - 4)$ → $x = \dfrac{a^3}{4}$

∴ Graph crosses non-vertical asymptote

d. Sketch the graph of $y = f(x)$ on the axes below, showing the asymptotes with their equations, the coordinates of the stationary point at $x = 0$ and the coordinates of any axes intercepts. Do **not** attempt to determine the coordinates of any other stationary points. 3 marks