4.2.2 Properties of the Definite Integral

When we defined the integral

$$\int_a^b f(x)dx,$$

we have so far defined this when $a < b$. However, if $b < a$, the Riemann Sum definition still works, but Δx changes from $\frac{b-a}{n}$ to $\frac{a-b}{n} = \frac{-(b-a)}{n}$. Thus,

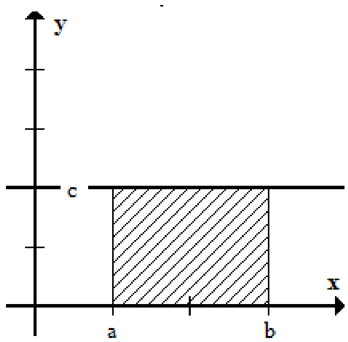
$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Further, if $a = b$, then $\Delta x = \frac{b-b}{n} = 0$, and as such

$$\int_b^b f(x)\, dx = 0.$$

Further, to help evaluate a definite integral, we have the following four properties, as long as f and g are continuous functions:

1. $\displaystyle\int_a^b c\, dx = c(b-a)$, for any constant c



2. $\displaystyle\int_a^b (f(x)+g(x))\ dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$

3. $\displaystyle\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$ for any constant c

4. $\displaystyle\int_a^b (f(x)-g(x))\ dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

Proving these is a quite simple task. Property 1 follows from the fact that we are integrating a constant height, and our figure would just be a rectangle – area of that is height c times width $b-a$. Properties 2, 3 and 4 are all proven similarly, and we prove property 3 here:

$$\begin{aligned}\int_a^b cf(x)dx &= \lim_{n\to\infty}\sum_{i=1}^{n}(cf(x_i))\,\Delta x\\ &= \lim_{n\to\infty} c\cdot\sum_{i=1}^{n} f(x_i)\Delta x\\ &= c\cdot\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x\\ &= c\cdot\int_a^b f(x)dx\end{aligned}$$

Thus, a constant, but ONLY a constant can be pulled out in front of an integral sign.

Example 4.14. Use the above properties to find

$$\int_2^7 3 - 6x^2 dx.$$

Using the difference property, we have

$$\int_2^7 3 - 6x^2 \, dx = \int_2^7 3 \, dx - \int_2^7 6x^2 \, dx$$

By property 1, we have

$$\int_2^7 3 \, dx = 3(7-2) = 15.$$

Also,

$$\int_2^7 6x^2 \, dx = 6 \int_2^7 x^2 \, dx,$$

but to evaluate $\int_2^7 x^2 dx$, we need to treat this as a Riemann Sum, with $\Delta x = \dfrac{7-2}{n} = \dfrac{5}{n}$ and $x_i = 2 + \dfrac{5i}{n}$

$$\begin{aligned}
\int_2^7 x^2 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(2 + \frac{5i}{n}\right)^2 \cdot \frac{5}{n} \\
&= \lim_{n\to\infty} \frac{5}{n} \sum_{i=1}^{n} 4 + \frac{20i}{n} + \frac{25i^2}{n^2} \\
&= \lim_{n\to\infty} \frac{5}{n} \left[\sum_{i=1}^{n} 4 + \sum_{i=1}^{n} \frac{20i}{n} + \sum_{i=1}^{n} \frac{25i^2}{n^2}\right] \\
&= \lim_{n\to\infty} \frac{5}{n} \left(4n + \frac{20}{n} \cdot \frac{n(n+1)}{2} + \frac{25}{n^2} \cdot \frac{n(n+1)(2n+1)}{6}\right) \\
&= \lim_{n\to\infty} \frac{5}{n} \left(4n + 10(n+1) + \frac{25(n+1)(2n+1)}{6n}\right) \\
&= 5 \cdot \lim_{n\to\infty} 4 + 10 \cdot \frac{n+1}{n} + 25 \cdot \frac{2n^2+3n+1}{6n^2} \\
&= 5 \cdot \left(4 + 10 + \frac{25}{3}\right) \\
&= \frac{335}{3}
\end{aligned}$$

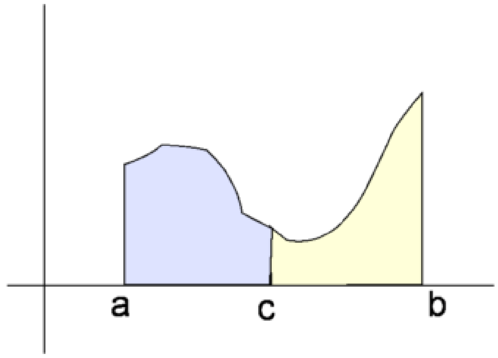
Thus, we have

$$\int_2^7 3 - 6x^2 dx = 15 - 6 \cdot \left(\frac{335}{3}\right) = 15 - 670 = -655.$$

A fifth property involves combining the bounds on two integrals over the same function:

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx,$$

which can easily be seen by the picture below:



After all, if we want to find the area under $f(x)$ from a to b, we can split it somewhere in the middle and add those two areas.

Example 4.15. Suppose that

$$\int_2^{10} f(x)\,dx = 13 \qquad \text{and} \qquad \int_2^7 f(x)\,dx = -3.$$

Find

$$\int_7^{10} f(x)l dx.$$

The answer here lies in a simple application from the following equation:

$$\int_2^{10} f(x)\,dx = \int_2^7 f(x)\,dx + \int_7^{10} f(x)\,dx$$

$$13 = -3 + \int_7^{10} f(x)\,dx$$

Therefore,

$$\int_7^{10} f(x)\,dx = 16$$

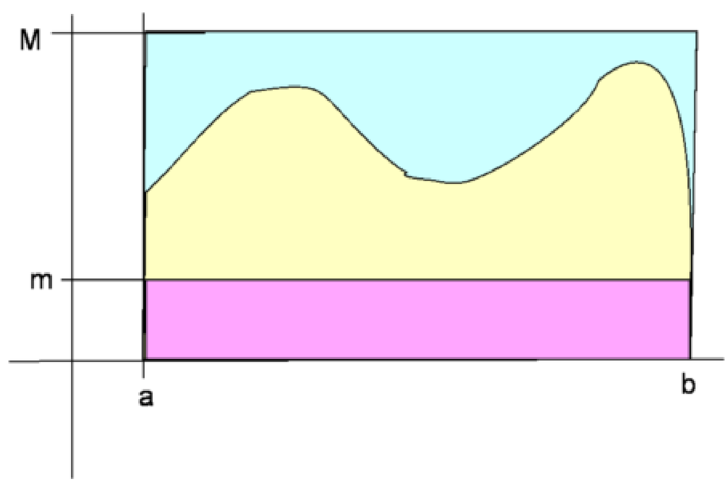
This property is true regardless if $a < c < b$ or not. It doesn't matter at all. Any other configuration would simply be a rearrangement of the variables. However, the following three properties are only true for $a < b$:

6) If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq 0$

7) If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

8) If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$.

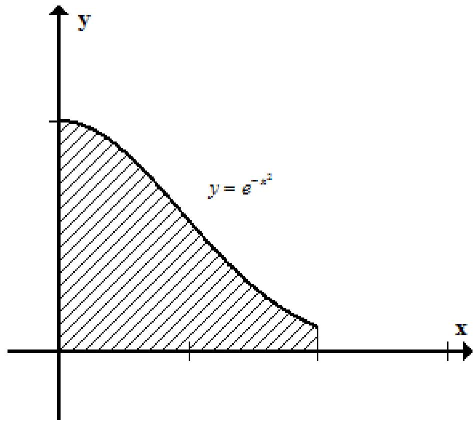
Property 6 implies a positive function gives positive areas. Property 7 says that a bigger (higher) function will have larger area. Duh. Property 8 states that is a function is trapped between two horizontal lines, and as such, the area under the curve is trapped between the areas of two rectangles. None of these really need to be proven, but we we can demonstrate pictorially:



Example 4.16. Estimate the value of

$$\int_0^2 e^{-x^2}\,dx.$$

Take a look at the graph,



From the graph, you can tell that it is a DECREASING function. You can also find this out by the first derivative test. Unfortunately, we do not learn the derivative of exponential functions until next semester.

You can see from the graph, our function has the smallest y value at $x = 2$, which is e^{-4}. It has its highest y-value at $x = 0$, which is $e^0 = 1$. Therefore,

$$e^{-4} < e^{-x^2} < e^0 = 1.$$

Thus, by property 8, we have

$$e^{-4}(2-0) \leq \int_0^2 e^{-x^2} dx \leq 1(2-0)$$

which gives

$$2e^{-4} \leq \int_0^2 e^{-x^2} dx \leq 2.$$