VCE, VCAL and VET

Victorian Certificate of Education Victorian Certificate of Applied Learning Vocational Education and Training

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Supplement 2

Mathematical Methods (CAS) study advice

This advice has been developed in response to queries from teachers during the first year of full cohort implementation of Mathematical Methods (CAS) Units 3 and 4 in 2010, and the VCAA metropolitan and regional workshops for first time teachers of Mathematical Methods (CAS) Units 3 and 4 conducted in 2010.

Strictly increasing and strictly decreasing functions

Students should be familiar with the notions of *strictly increasing* and *strictly decreasing* for functions of a single real variable, and be able to identify intervals over which such a function is strictly increasing or strictly decreasing. An example relating to the interval over which a simple polynomial function is strictly decreasing was included as Question 4 of the sample questions for Mathematical Methods (CAS) Examination 1, published on the VCAA website in early 2010: www.vcaa.vic.edu.au/vcaa/vce/studies/mathematics/cas/publications/mmcas1-sampfin-w.pdf

Strictly increasing

A function f is said to be *strictly increasing* when $a < b$ implies $f(a) < f(b)$ for all a and b in its domain. This definition does not require f to be differentiable, or to have a non-zero derivative, for all elements of the domain. If a function is strictly increasing, then it is a one-to-one function and has an inverse function that is also strictly increasing.

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VICTORIAN CURRICULUM AND ASSESSMENT AUTHORITY

Example 1

The function $f: R \to R, f(x) = x^3$ is strictly increasing with zero gradient at the origin, as illustrated in Figure 1:

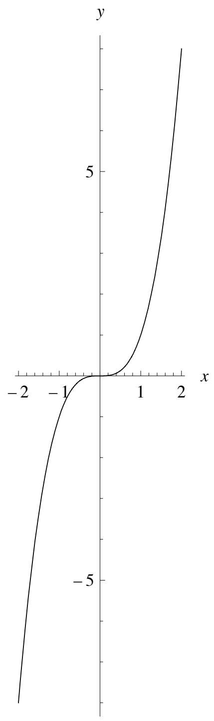

Figure 1: Part of the graph of *f*

The inverse function $f^{-1}: R \to R, f(x) = x^{\frac{1}{3}}$ is also strictly increasing, with a vertical tangent of undefined gradient at the origin.

Example 2

The hybrid function *g* with domain [0, ∞) and rule:

$$g(x) = \begin{cases} x^2 & 0 \le x \le 2 \\ 2x & x > 2 \end{cases}$$

is strictly increasing, and is not differentiable at $x = 2$, as illustrated in Figure 2:

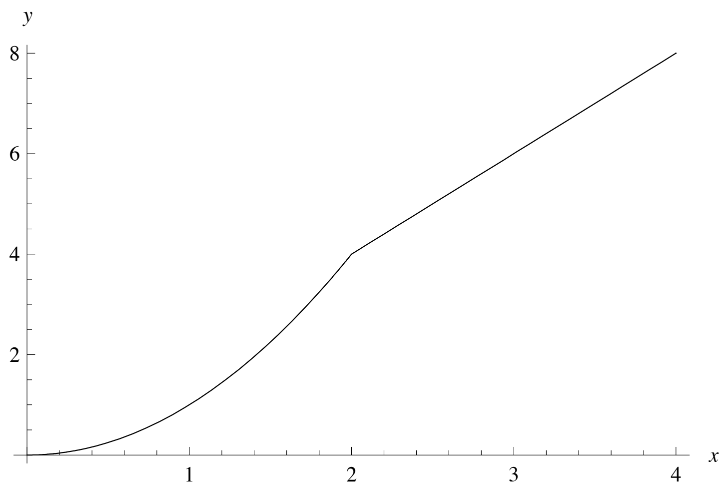

Figure 2: Part of the graph of *g*

A function is said to be strictly increasing over an interval when $a < b$ implies $f(a) < f(b)$ for all a and b in the interval. Thus, the function $h : R \rightarrow R$, $h(x) = |x| - x^3$ is not strictly increasing, as illustrated in Figure 3:

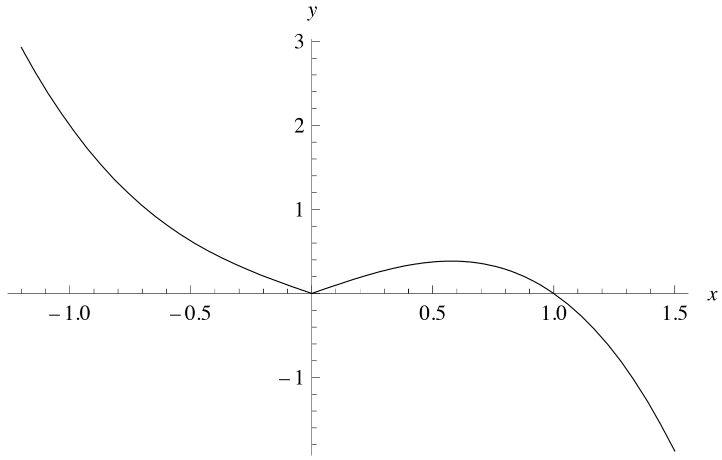

Figure 3: Part of the graph of h

However, the function is strictly increasing over the *interval* $\left[0, \frac{1}{\sqrt{3}}\right]$, as illustrated in Figure 4:

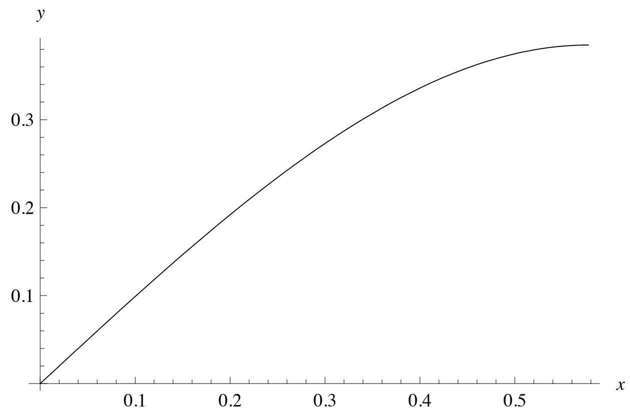

Figure 4: Interval over which h is strictly increasing

Strictly decreasing

A function f is said to be *strictly decreasing* when $a < b$ implies $f(a) > f(b)$ for all a and b in its domain. A function is said to be strictly decreasing over an interval when $a < b$ implies $f(a) > f(b)$ for all a and b in the interval.

Example 3

The function $f : R \rightarrow R, f(x) = \dfrac{1}{e^x + 1}$ is strictly decreasing over R, as illustrated in Figure 5:

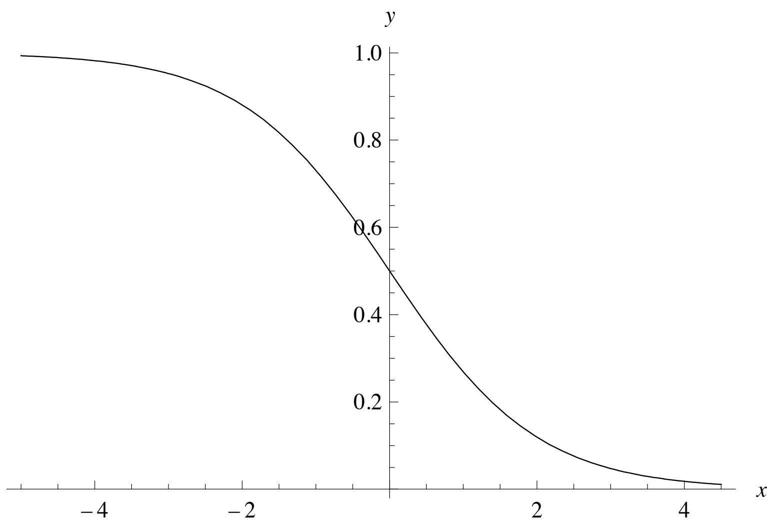

Figure 5: Part of the graph of f

Example 4

The function $g : R \to R$, $g(x) = \cos(x)$ is not strictly decreasing as illustrated in Figure 6:

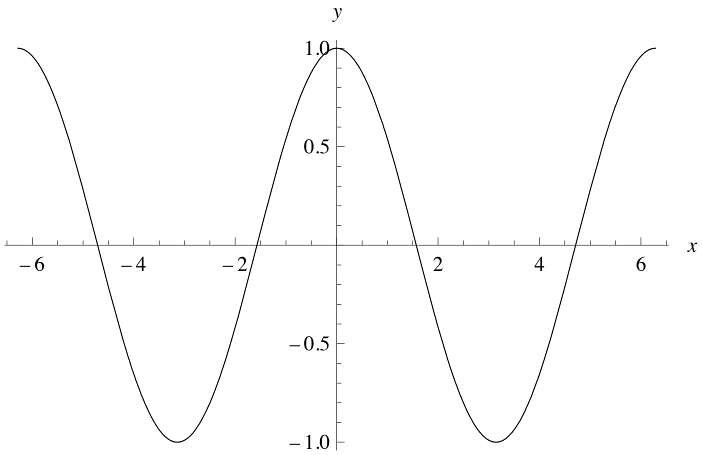

Figure 6: Part of the graph of g

However, g is strictly decreasing over the interval $[0, \pi]$ as illustrated in Figure 7:

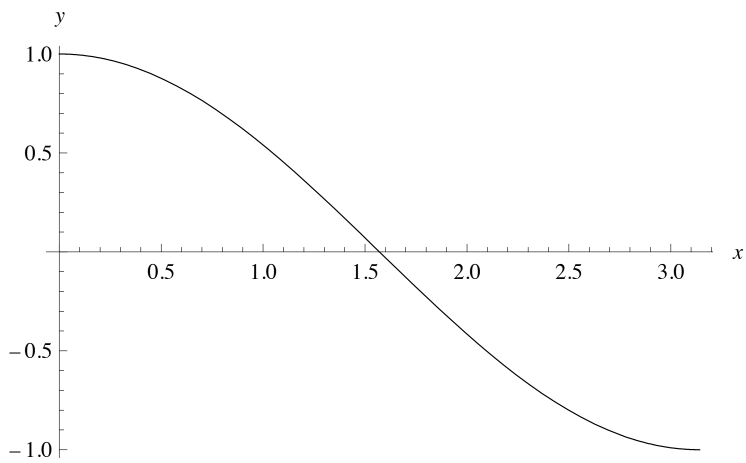

Figure 7: The graph of g on the interval $[0, \pi]$

If a function with rule $y = f(x)$ is strictly increasing over an interval then the function with rule $y = -f(x)$ will be strictly decreasing on the same interval. For example $y = \sin(x)$ is strictly increasing on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $y = -\sin(x)$ is strictly decreasing over this interval.

Maximum and minimum problems in modelling contexts

Functions are used to model various situations in theoretical and practical contexts, and to carry out related analysis including the identification of local and global maximum and minimum values. In such contexts the domain and graph of a function are important aspects of analysis. In some cases the analysis can be simplified using a related function. A differentiable function may or may not have a maximum value or minimum value over a given interval, and any maximum value or minimum value, where this exists, can be identified by considering zeros of the derivative or endpoints. Depending on whether exact or approximate answers are required, analytical, graphical or numerical methods can be employed, as applicable, to determine the required values.

Example 5

Let d be the function that models the distance from the origin to any point on the graph of $y = \frac{4}{x}$ where $x > 0$, as illustrated in Figure 8:

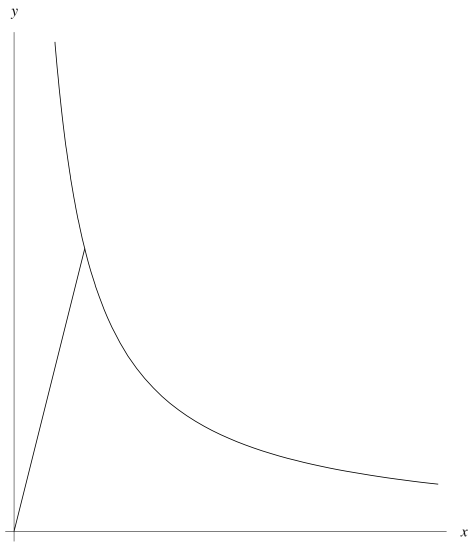

Figure 8: Part of the graph of $y = \frac{4}{x}$ showing a line segment from the origin to a point on the curve

Consideration of the geometry of the situation indicates that d will have a minimum value but no maximum value. This function can be defined explicitly $d : R^+ \to R,\ d\,(x) = \sqrt{x^2 + \frac{16}{x^2}}$. The minimum value for the distance will occur when the minimum value of the square of the distance also occurs. That is when $2x - \frac{32}{x^3} = 0$ for $x > 0$, which gives $d_{\min} = \sqrt{8}$ when $x = 2$. In measurement contexts where distance or length is involved, the modelling function is often of the form $f(x) = \sqrt{u(x)}$ and its derivative is of the form $f'(x) = \frac{u'(x)}{2\sqrt{u(x)}}$ with zeros occurring when $u'(x) = 0$ and $u(x) > 0$. Question 12 from Mathematical Methods (CAS) Examination 1 in 2007 and Question 3 from Section 2 of Mathematical Methods (CAS) Examination 2 in 2010 are related types of problems. In the latter case the modelling function for volume has a single maximum value which is a local maximum, but no minimum value as shown in Figure 9:

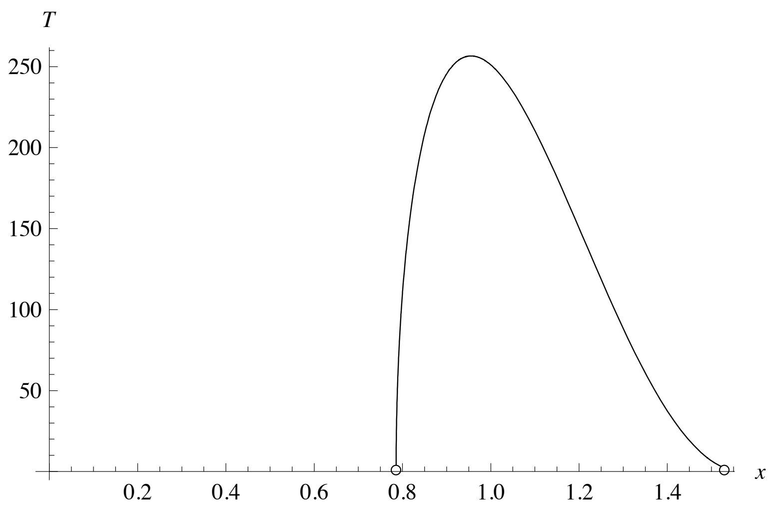

Figure 9: The graph of $T : \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \to R,\ T(x) = \frac{4000}{3}\sqrt{\cos^4(x) - 2\cos^6(x)}$

Example 6

Let A be the function that models the total enclosed area when a 100cm piece of wire is cut into two pieces, where one piece is used to form the perimeter of a square, and the other piece is used to form the circumference of a circle. Consideration of the geometry of the situation indicates that A will have both a minimum value and a maximum value. This function can be defined explicitly and is readily recognisable as a quadratic function $A:[0, 100] \rightarrow R, A(x) = \frac{x^2}{16} + \frac{(100-x)^2}{4\pi}$ where x cm is the length of the piece of wire used to form the perimeter of the square. Figure 10 shows that the local minimum corresponds to the minimum value of A, and the left endpoint of the domain at $x = 0$ corresponds to the maximum value of A:

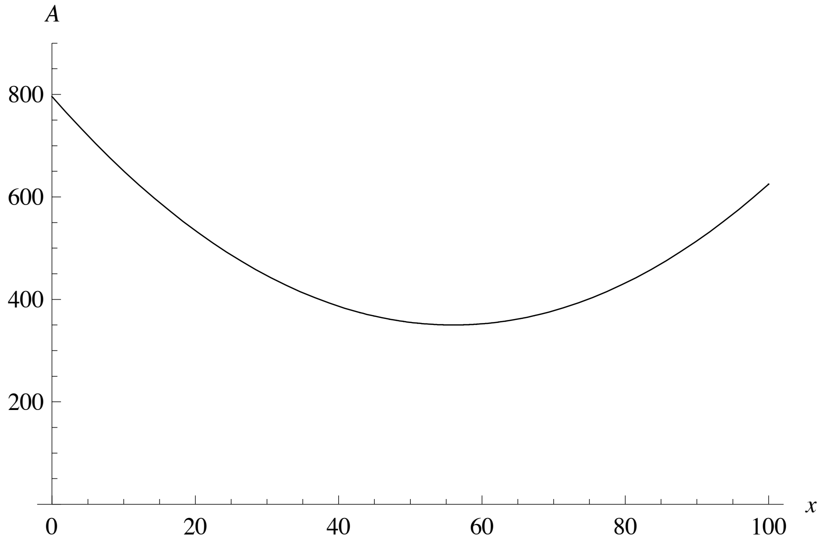

Figure 10: The graph of $A:[0, 100] \rightarrow R, A(x) = \frac{x^2}{16} + \frac{(100-x)^2}{4\pi}$

Question 1 from Section 2 of Mathematical Methods (CAS) Examination 2 in 2007 and Question 3 from Section 2 of Mathematical Methods(CAS) Examination 2 in 2008 are related types of problems.

Example 7

The function $f: R \rightarrow R, f(x) = \frac{1}{e^x + 1}$ in Example 3 has no maximum value and no minimum value. The graph of f is asymptotic to $y = 1$ from below as $x \rightarrow -\infty$ and asymptotic to $y = 0$ from above as $x \rightarrow \infty$.

Student responses to tasks involving questions such as these (or variations and generalisations) where problem-solving, modelling or investigative techniques or approaches are required, provide them with the opportunity to demonstrate achievement of aspects of all three outcomes. In particular, where CAS technology is used, aspects of Outcome 3 can be addressed. As stated on page 135 of the study design, for Outcome 3 students should be to be able to:

> select and appropriately use a computer algebra system and other technology to develop mathematical ideas, produce results and carry out analysis in situations requiring problem-solving, modelling or investigative techniques or approaches.

To achieve this outcome, students will draw on knowledge and skills outlined in all the areas of study. This includes key knowledge and key skills as specified on pages 135–136 of the study design:

Key knowledge

This knowledge includes:

- exact and approximate specification of mathematical information such as numerical data, graphical forms and general or specific forms of solutions of equations produced by use of a computer algebra system;
- domain and range requirements for a computer algebra system's specification of graphs of functions and relations;
- the role of parameters in specifying general forms of functions and equations;
- the relation between numerical, graphical and symbolic forms of information about functions and equations and the corresponding features of those functions or equations;
- the similarities and differences between formal mathematical expressions and their computer algebra system representation;
- the appropriate selection of a technology application, in particular, computer algebra systems, in a variety of mathematical contexts.

Key skills

These skills include the ability to:

- distinguish between exact and approximate presentations of mathematical results produced by a computer algebra system, and interpret these results to a specified degree of accuracy;
- produce results using a computer algebra system which identify examples or counter-examples for propositions;
- produce tables of values, families of graphs or collections of other results using a computer algebra system, which support general analysis in problem-solving, investigative or modelling contexts;
- use appropriate domain and range specifications to illustrate key features of graphs of functions and relations;
- identify the relation between numerical, graphical and symbolic forms of information about functions and equations and the corresponding features of those functions or equations;
- specify the similarities and differences between formal mathematical expressions and their computer algebra system representation, in particular, equivalent forms of symbolic expressions;
- make appropriate selections for a computer algebra system and other technology applications in a variety of mathematical contexts, and provide a rationale for these selections;
- relate the results from a particular application to the nature of a particular mathematical task (investigative, problem solving or modelling) and verify these results;
- specify the process used to develop a solution to a problem using a computer algebra system, and communicate the key stages of mathematical reasoning (formulation, solution, interpretation) used in this process.

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