Worksheet 4.4 Applications of Integration

Section 1 Movement

Recall that the derivative of a function tells us about its slope. What does the slope represent? It is the change in one variable with respect to the other variable. Say a line has a constant slope of 4; then for every 1 unit change in x, there will be a 4 unit change in y. Say we had a function that represented the movement of a car, so that the distance was plotted as a function of time. The change in distance over a small amount of time would represent the speed of the car. Thus in this case the slope of the function represents the speed of the car, and is given by $\frac{dx}{dt}$. The rate of change in speed of the car is called acceleration and this is given by $\frac{d}{dt}(\frac{dx}{dt}) = \frac{d^2x}{dt^2}$.

So if we have a function $x = f(t)$ that represents distance as a function of time, then $\frac{dx}{dt}$ is the speed and $\frac{d^2x}{dt^2}$ is the acceleration. Conversely, if we have a function that represents the velocity of a vehicle and we integrate it we get the distance travelled as a function of time.

The methods of integration and differentiation can be used to solve problems involving movement.

Example 1 : If the velocity of a particle is given by $v = 3t^2$, what is the distance travelled as a function of time? Since $v = \frac{dx}{dt}$, where x is the distance, the anti derivative of v will tell us the distance.

$$x = \frac{3t^3}{3} + c = t^3 + c$$

Example 2 : If the velocity of a particle is given as $v = 3t^2$ metres per second, what is the distance travelled between $t = 0$ and $t = 2$? In example 1, we worked out that the distance travelled was $x = t^3 + c$. Therefore

$$\text{Distance } = (t^3 + c)\Big]_0^2 = (2^3 + c) - (0^3 + c) = 8 \text{ metres}$$

The particle covers a distance of 8 metres between $t = 0$ and $t = 2$.

Section 2 Initial-Value Problems

Recall that, when working out anti derivative problems, there is a constant of integration that is undetermined (and which we have usually denoted by c). Initial-value problems ask us to

find anti derivatives which take on specific values at certain points so that we can determine the value of the constant.

Example 1 : If $\frac{dx}{dt} = 5$ and $x = 9$ when $t = 0$, what is x as a function of time? As $\frac{dx}{dt} = 5$, then $x = 5t + c$. Using the information that when $t = 0$, $x = 9$ we can now write an equation to solve for c:

$$9 = 5 \times 0 + c$$

which has the solution $c = 9$. The complete solution for x is $x = 5t + 9$.

Example 2 : The acceleration of a car is given by $a = 6t$ and the velocity is 2 when $t = 0$, and the distance from home is 1 when $t = 0$. What is the distance from home as a function of time? Acceleration is the derivative of velocity, so velocity is the anti derivative of acceleration. Then

$$v = \int 6t\, dt = 3t^2 + c_1$$

When $t = 0$, $v = 2$, so that we can write down an equation for c_1 and solve it: $2 = 0 + c_1$. Therefore, $c_1 = 2$. The velocity is then $v = 3t^2 + 2$. Distance is the anti derivative of velocity, which gives

$$x = \int (3t^2 + 2)\, dt = t^3 + 2t + c_2$$

Using $x = 1$ when $t = 0$, we can write down an equation for c_2: $1 = 0 + 0 + c_2$. Therefore $c_2 = 1$, and so the distance as a function of time is

$$x = t^3 + 2t + 1$$

Example 3 : If $\frac{d^2x}{dt^2} = 3$, and when $t = 0$ we have $\frac{dx}{dt} = 0$ and $x = 0$, what is x as a function of t?

$$\begin{aligned} \frac{d^2x}{dt^2} &= 3 \\ \frac{dx}{dt} &= 3t + c \end{aligned}$$

Using the information we are given for $\frac{dx}{dt}$ at $t = 0$, we find $3 \times 0 + c = 0$ so that $c = 0$. Then $\frac{dx}{dt} = 3t$ for all t. The anti derivative of this will give us x:

$$x = \frac{3t^2}{2} + c$$

Using the information for x at $t = 0$, we find $c = 0$, so that $x = \frac{3t^2}{2}$ for all t.

Section 3 Application to Growth

Exponential functions are used to represent the growth and decay of populations and radioactive elements, among other things. We can use a general form of an equation for exponential growth or decay and we find a specific equation which uses initial values as in the application of integration to motion.

Exponential growth and decay is represented by the equation $P(t) = P(0)e^{kt}$ where $P(t)$ is the population at time t, $P(0)$ is the population at $t = 0$, and k is some constant which depends on the population being looked at. A similar formula applies to the decay of radioactive material. $P(0)$ would then represent the amount of radioactive material at $t = 0$.

Example 1 : What is $P(10)$ if $P(0) = 100$ and $k = 1$ given $P(t) = P(0)e^{kt}$.

$$\begin{aligned} P(10) &= P(0)e^{kt} \\ &= 100e^{10} \end{aligned}$$

Example 2 : If $P(10) = 1000$ and $P(0) = 100$, what is k in the expression $P(t) = P(0)e^{kt}$? We put $t = 10$ into the equation for $P(t)$ and equate this to what we are given at $P(10)$.

$$P(10) = 1000 = 100e^{k10}$$

This equation can be solved for k:

$$\begin{aligned} 1000 &= 100e^{10k} \\ 10 &= e^{10k} \\ \log 10 &= 10k \\ k &= \frac{1}{10}\log 10 \end{aligned}$$

Example 3 : If the growth constant for a population of bees is $\frac{1}{10}$ and the initial population of a hive is 75, what is the population at time t?

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ &= 75e^{\frac{1}{10}t} \end{aligned}$$

Notice that if $k > 0$ the population is growing but if $k < 0$ the population is getting smaller.

Example 4 : For what values of k does the population $P(t) = P(0)e^{kt}$ remain constant? We need $P(t) = P(0)$ for all t. Then

$$\begin{aligned} P(t) &= P(0)e^{kt} \\ 1 &= e^{kt} \end{aligned}$$

This is true when $k = 0$.