Starting Point 3
Disappearing Wombats

Contents

Introduction

Modelling is often used to predict population growth. One model which considers birth and death rates and another which also considers food supply can be used to predict the population of wombats on a remote island t years after the initial 200 wombats were setlled there.

Model One	**Model Two**
$\frac{dW}{dt} = (m-n)W$	$\frac{dW}{dt} = (m-n-kW)W$

To compare these it is necessary to find an expression for both of them with W in terms of t, thereby allowing us to predict the population and explore the effects of m, n and k on the population growth.

Variables and Constants

m = positive constant related to birth rate
n = positive constant related to death rate
k = positive constant related to food supply
t = time (in years)
W = number of wombats

Assumptions

- All factors influencing birth, death and food supply (eg. disease, inferitlity) are taken into account by m, n and k.
- No natural disasters occur throughout the course of the wombats' lives.
- Since the models are continuous and the number of wombats is discrete, W values are rounded down to the nearest whole wombat.
- There were no wombats on the island before the initial 200 were settled there.
- The island is free from human interference

Model One

A general expression for population (W) in terms of t, will enable us to explore the effects of m and n on our model. Antidifferentiating $\frac{dW}{dt} = (m-n)W$ will give us this expression.

$$\frac{dW}{dt} = (m-n)W$$

$$\therefore \frac{dt}{dW} = \frac{1}{(m-n)W}$$

$$t = \int \frac{1}{(m-n)W}\, dW$$

$$= \frac{1}{(m-n)} \times \int \frac{1}{W}\, dW$$

$$= \frac{1}{(m-n)} \times \log_e W + c \qquad (c \text{ is a constant})$$

Transposing to make W the subject:

$$t - c = \frac{\log_e W}{(m - n)}$$

$$(t - c)(m - n) = \log_e W$$

$$W = e^{(t-c)(m-n)}$$

$$= e^{(m-n)t-(m-n)c}$$

$$= e^{(m-n)t} \times e^{-(m-n)c}$$

$$W = Ae^{(m-n)t} \qquad \left(\text{where } A = e^{-(m-n)c}\right)$$

To evaluate A:

$t = 0$, $W = W_i$ (where W_i = initial population):

$$W = Ae^{(m-n)t}$$

$\therefore$ when $t = 0$:

$$W_i = Ae^{0\times(m-n)}$$

$$= Ae^0$$

$$= A$$

$W_i = A = 200$, so:

$$\boxed{W = 200e^{(m-n)t}}$$

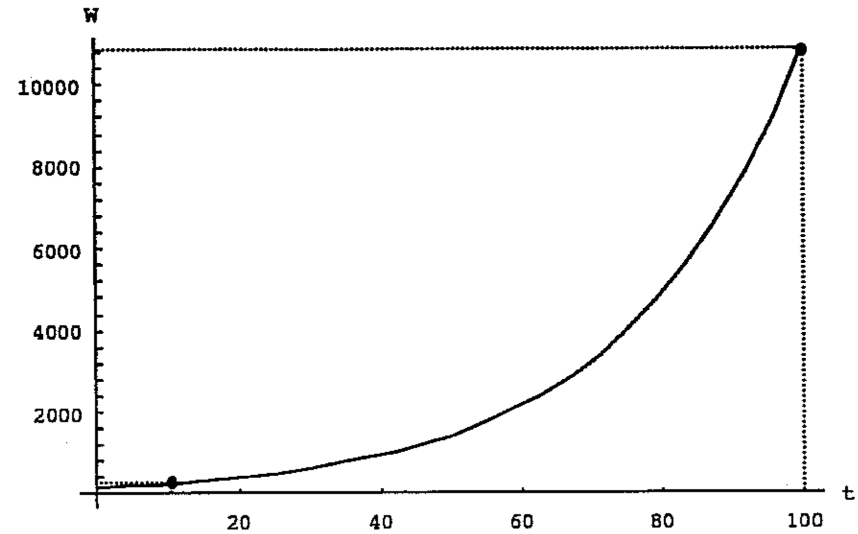
We can check this general equation by substituting two values for m and n (0.1 and 0.06 respectively) from the beginning, performing the same calculations and comparing the results.[1]

The expression $W = 200e^{0.04t}$ can be used to find the population for different values of t:

t = 10 years	t = 100 years
$W = 200e^{0.04t}$ $= 200e^{0.04\times10}$ $= 200 \times 1.492$ $= 298.365$ ≈ 298	$W = 200e^{0.04t}$ $= 200e^{0.04\times100}$ $= 200 \times 54.598$ $= 10919.630$ ≈ 10919

1 See Appendix A for checking.

The graph of $W = 200e^{0.04t}$: 2



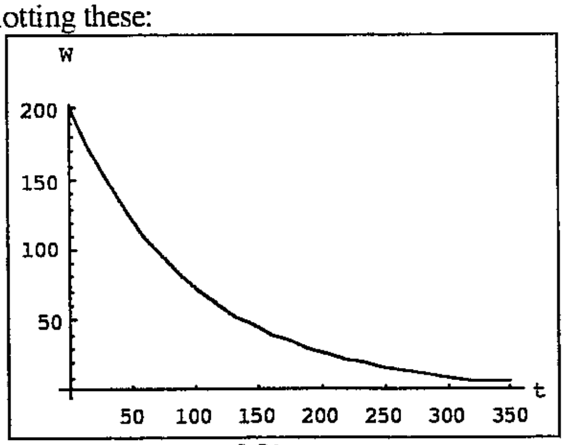
It is seen that the rate of growth is rapid and although it would be reasonable to expect that the population would be greater at $t = 100$ and $t = 10$, an increase by a factor of approximately 36 seems unrealistic.

Matters of Life and Death

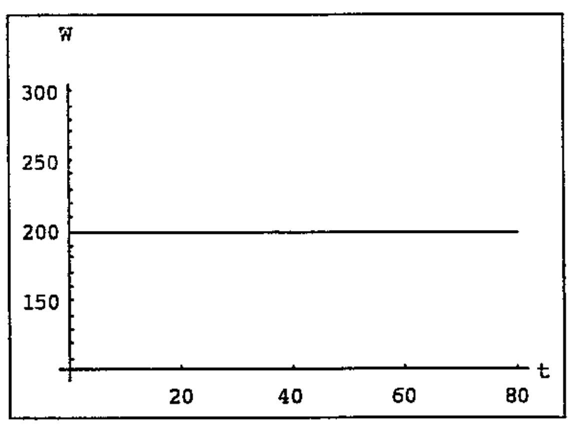
To further explore the effects of m (between 0.08 and 0.12) and n (between 0.06 and 0.09), we look at the three possible situations: $m < n$, $m = n$ or $m > n$ and choose values of m and n which will create these situations:

	m	n	Equation
$m < n$	0.08	0.09	$W = 200e^{t(0.08-0.09)}$ $= 200e^{-0.01t}$
$m = n$	0.09	0.09	$W = 200e^{t(0.09-0.09)}$ $= 200e^{0t} = 200$
$m > n$	0.12	0.06	$W = 200e^{t(0.12-0.06)}$ $= 200e^{0.06t}$

Plotting these:

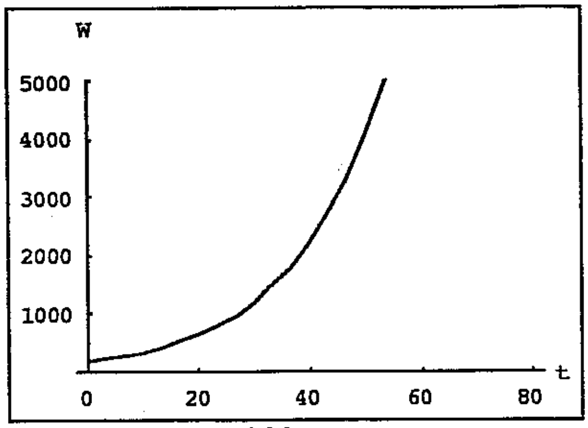


$W = 200e^{-0.01t}$ $(m < n)$



$W = 200$ $(m = n)$

2 Graphed with *Mathematica: Student Version*, Wolfram Research Inc, Champaign, 1993.

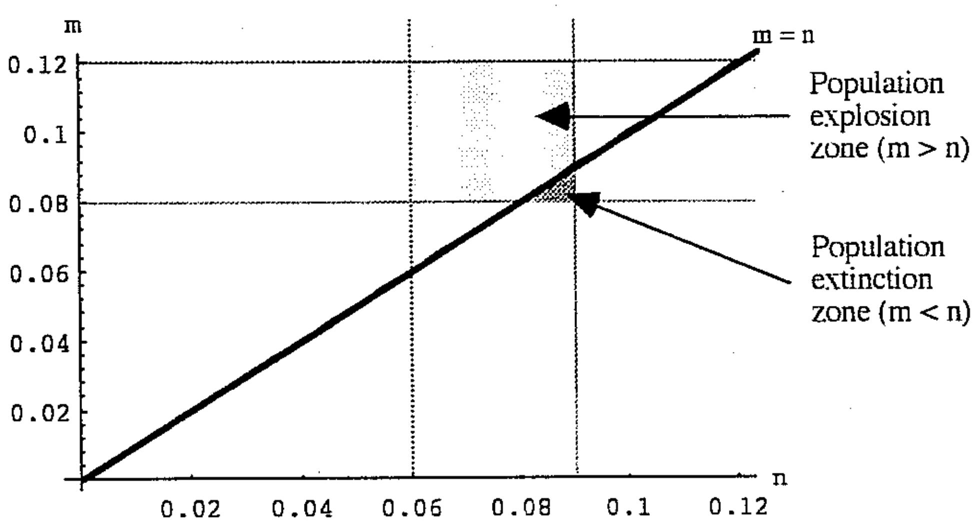


$W = 200e^{0.06t}$ $(m > n)$

It is seen that when $m < n$, the curve decays exponentially and the population eventually dies out. This is true because $(m - n)$ will be negative, resulting in an exponential function raised to a negative power. Physically, this refers to the point where the birth rate is lower than the death rate, resulting in a decreasing population and eventually extinction.

When $m = n$, the curve becomes a horizontal line (ie. no growth) because $m - n = 0$, resulting in the function: $W = Ae^{0t} = A = W_i$. This will occur when the birth and death rates are equal. When $m > n$ however, the curve is typical of exponential growth because $m - n > 0$. In reality this means that the growth rate is greater than the death rate, resulting in population explosion.

This can be summarised below:



Limitations

This model is unrealistic because it does not take the food supply, habitable area and other elements necessary to sustain wombat life into account. In reality, unlimited growth (eg. when $m > n$) will never occur, instead it would be expected that after a certain time the island would not have the resources to support more life. A better model should incorporate a negative effect on the rate of growth casued by over population.

Model Two

A new model which is given by $\frac{dW}{dt} = (m - n - kW)W$ and also considers the limitations of food supply (k) should prove much more accurate. It can be seen that as k increases, the value of kW also increases, $(m - n - kW)$ decreases and hence, the rate of wombat reproduction decreases. The opposite is true when $k \to 0$ (ie. $\frac{dW}{dt} \to +\infty$). This indicates an inverse relationship between k and the amount of food (ie. Amount of food $\propto \frac{1}{k}$).

If we take $m = 0.10$, $n = 0.06$ (as for Model One) and $k = 0.00005$, then finding an expression for W in terms of t will help us compare the two models. To find this expression, however, $\frac{dW}{dt} = (m - n - kW)W$ must be solved.

Substituting m, n, and k:

$$\frac{dW}{dt} = (0.10 - 0.06 - 0.00005W)W$$

$$= (0.04 - 0.00005W)W$$

Inverting:

$$\frac{dt}{dW} = \frac{1}{(0.04 - 0.00005W)W}$$

Using partial fractions:

$$\frac{dt}{dW} = \frac{1}{(0.04 - 0.00005W)W}$$

$$= \frac{T}{(0.04 - 0.00005W)} + \frac{V}{W} \quad \text{(where T and V are constants)}$$

Matching up the co-efficients:

$$1 = TW + V(0.04 - 0.00005W)$$

when $W = 0$,

$$1 = T(0) + V(0.04 - 0.00005(0))$$

$$= 0.04V$$

$$\therefore \frac{1}{0.04} = V = 25$$

and when $W = 1$,

$$1 = T(1) + 25(0.04 - 0.00005(1))$$

$$= T + 25 \times 0.03995$$

Hence $T = 1 - 25 \times 0.03995$

$$= 1 - 0.99875 = 0.00125$$

Substituting T and V:

$$\frac{dt}{dW}=\frac{0.00125}{(0.04-0.00005W)}+\frac{25}{W}$$

$$=\frac{0.00125}{(0.04-0.00005W)}\times\frac{20000}{20000}+\frac{25}{W}$$

$$=\frac{25}{800-W}+\frac{25}{W}$$

Antidifferentiating:

$$t=\int\frac{25}{800-W}\,dW+\int\frac{25}{W}\,dW$$

$$=-25\int\frac{1}{W-800}\,dW+25\int\frac{1}{W}\,dW$$

$$=25\log_e W-25\log_e(W-800)+c$$

$$=25\log_e\left(\frac{W}{W-800}\right)+c$$

Transposing to make W the subject:

$$\log_e\left(\frac{W}{W-800}\right)=\frac{t-c}{25}$$

$$\frac{W}{W-800}=e^{\frac{t-c}{25}}$$

$$=e^{\frac{t}{25}}\times e^{\frac{-c}{25}}$$

$$=Be^{\frac{t}{25}} \qquad (\text{where } B=e^{\frac{-c}{25}})$$

$$W=Be^{\frac{t}{25}}\times(W-800)$$

$$W-WBe^{\frac{t}{25}}=-800Be^{\frac{t}{25}}$$

$$W\left(1-Be^{\frac{t}{25}}\right)=-800Be^{\frac{t}{25}}$$

$$\therefore\ \mathrm{W}=\frac{-800Be^{\frac{t}{25}}}{1-Be^{\frac{t}{25}}}$$

Simplifying:

$$W = \frac{-800Be^{\frac{t}{25}}}{1 - Be^{\frac{t}{25}}}$$

$$= \frac{-800}{Be^{-0.04t} - 1}$$

$$= \frac{800}{1 - Ge^{-0.04t}} \qquad \left(G = B^{-1}\right)$$

when $t = 0$, $W = 200$. Therefore:

$$200 = \frac{800}{1 - Ge^{-0.04 \times 0}},$$

$$= \frac{800}{1 - G}$$

$$200 - 200G = 800$$

$$-200G = 600$$

$$G = -3$$

Substituting G:

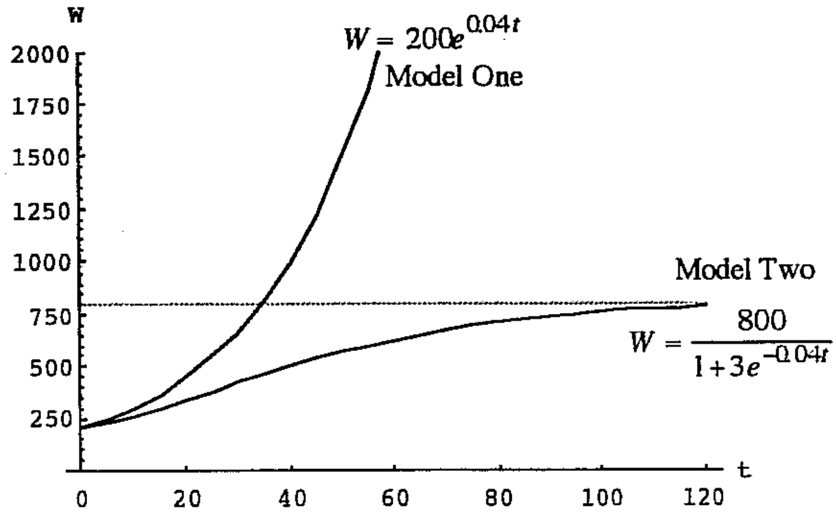
$$\boxed{W = \frac{800}{1 + 3e^{-0.04t}}, \quad t \geq 0}$$

Calculating W when $t = 10$ and 100 (as for Model One) will enable further comparison:

$t = 10$ years	$t = 100$ years
$W = \frac{800}{1 + 3e^{-0.04 \times 10}}$ $= \frac{800}{1 + 3e^{-0.4}}$ $= \frac{800}{1 + 2.012}$ $= 265.70$ ≈ 265	$W = \frac{800}{1 + 3e^{-0.04 \times 100}}$ $= \frac{800}{1 + 3e^{-4}}$ $= \frac{800}{1 + 0.0549}$ $= 758.33$ ≈ 758

For the same values of 10 and 100 years, the second model's predictions are far more conservative (11.1% and 93.1% more for $t = 10$ and 100 respectively). This would be due to k's influence.

Graphing this function against the Model One function with identical m and n values::



It is seen that the second model eventually reaches an asymptote (which would represent the maximum number of wombats the island could support). It follows that this 'ceiling value' is a direct result of k (and is explored later).

Another way of investigating Model Two would be finding the point when the rate of growth is maximum, ie. when $\frac{dW}{dt}$ is maximum. To do this we find when $\frac{d^2W}{dt^2} = 0$.

$$\frac{dW}{dt} = (m - n - kW)W$$

Therefore:

$$\begin{aligned}\frac{d^2W}{dt^2} &= \frac{d}{dt}\left(\frac{dW}{dt}\right)\\ &= (m-n-kW)W' + W(-kW')\\ &= (m-n-2kW)W'\end{aligned}$$

Since $\frac{dW}{dt} \neq 0$, $(m-n-2kW) = 0$. So:

$$m - n = 2kW$$

$$\therefore W = \frac{m-n}{2k}$$

To find the corresponding t value, substitute $W = \frac{m-n}{2k}$ into our original equation:

$$\frac{m-n}{2k} = \frac{800}{1+3e^{-0.04t}}$$

$$\frac{0.04}{0.0001} = \frac{800}{1+3e^{-0.04t}}$$

$$400 = \frac{800}{1+3e^{-0.04t}}$$

$$400 + 1200e^{-0.04t} = 800$$

$$1200e^{-0.04t} = 400$$

$$e^{-0.04t} = \frac{1}{3}$$

$$\begin{aligned}
-0.04t &= \log_e \frac{1}{3} \\
&= \log_e 1 - \log_e 3 \\
&= -\log_e 3 \\
t &= 25\log_e 3 \\
&\approx 27.47 \\
&\approx 27 \text{ years and } 5.6 \text{ months}
\end{aligned}$$

Proving it's a point of maximum:

when $W = 399$	when $W = 400$	when $W = 401$
$\frac{dW}{dt} = (m-n-kW)W$ $= (0.04-0.01995)399$ $= 7.99995$	$\frac{dW}{dt} = (m-n-kW)W$ $= (0.04-0.02)400$ $= 8$	$\frac{dW}{dt} = (m-n-kW)W$ $= (0.04-0.02005)401$ $= 7.99995$
	MAXIMUM	

Therefore at 27.47 years, $W = 400$ and the rate of reproduction is the greatest. This would be a good time to move wombats back to the mainland (to keep $W = 400$) so that the rate of reproduction will always be maximum. The rate can be calculated by substituting $\frac{m-n}{2k}$ into the formula $\frac{dW}{dt} = (m-n-kW)W$ as W. Therefore:

$$\begin{aligned}
\frac{dW}{dt} &= \frac{m(m-n)}{2k} - \frac{n(m-n)}{2k} - \frac{k(m-n)^2}{4k^2} \\
&= \frac{(m-n)^2}{2k} - \frac{(m-n)^2}{4k} \\
&= \frac{2(m-n)^2 - (m-n)^2}{4k} \\
&= \frac{(m-n)^2}{4k}
\end{aligned}$$

Substituting m, n and k:

$$\begin{aligned}
\left(\frac{dW}{dt}\right)_{max} &= \frac{(0.04)^2}{0.0002} \\
&= 8 \text{ wombats / year}
\end{aligned}$$

Therefore at the maximum rate, 8 wombats per year are born. This would be just before when food suppply would begin to seriously hinder population growth.

Food Glorious Food

To explore k's effect on the asymptote (ie. the population that the island can sustain) it is easier to solve a general equation for Model Two which gives W in terms of t, m, n, and k:

As $\dfrac{dW}{dt} = (m-n-kW)W$

$$\frac{dt}{dW} = \frac{1}{(m-n-kW)W}$$

$$= \frac{T}{(m-n-kW)} + \frac{V}{W} \quad \text{(where T and V are real constants)}$$

Matching up the co-efficients:

$$1 = TW + V(m-n-kW)$$

when $W = 0$,

$$1 = T(0) + V(m-n-k(0))$$

$$= V(m-n)$$

$$\therefore \frac{1}{m-n} = V$$

and when $W = 1$,

$$1 = T(1) + \frac{1}{m-n}(m-n-k(1))$$

$$= T + \frac{m-n-k}{m-n}$$

$$= T + \left(1 - \frac{k}{m-n}\right)$$

$$T = 1 - \left(1 - \frac{k}{m-n}\right)$$

$$= \frac{k}{m-n}$$

Substituting T and V:

$$\frac{dt}{dW} = \frac{\frac{k}{m-n}}{(m-n-kW)} + \frac{\frac{1}{m-n}}{W}$$

$$= \frac{k}{(m-n)(m-n-kW)} + \frac{1}{m-n}$$

Antidifferentiating:

$$t = \int \frac{k}{(m-n)(m-n-kW)}\, dW + \int \frac{1}{(m-n)W}\, dW$$

$$= \frac{k}{(m-n)} \int \frac{1}{(m-n-kW)}\, dW + \frac{1}{(m-n)} \int \frac{1}{W}\, dW$$

$$= \frac{k}{-k(m-n)} \log_e(m-n-kW) + \frac{1}{(m-n)} \log_e W + c$$

$$= \frac{1}{(m-n)} \log_e\left(\frac{W}{m-n-kW}\right) + c$$

Transposing:

If $t = \frac{1}{(m-n)} \log_e\left(\frac{W}{m-n-kW}\right) + c$ then,

$$(t-c)(m-n) = \log_e\left(\frac{W}{m-n-kW}\right)$$

$$e^{(t-c)(m-n)} = \frac{W}{m-n-kW}$$

$$e^{t(m-n)} \times e^{-c(m-n)} = \frac{W}{m-n-kW}$$

$$Ae^{t(m-n)} = \frac{W}{m-n-kW} \qquad \left(\text{where A } = e^{-c(m-n)}\right)$$

Let $X = Ae^{t(m-n)}$ for simplification:

$$X = \frac{W}{m-n-kw}$$

$$X(m-n-kw) = W$$

$$X(m-n) - kWX =$$

$$X(m-n) = W(1+kX)$$

$$\frac{X(m-n)}{1+kX} = W$$

$$\frac{m-n}{X^{-1}+k} = W$$

Substituting X:

$$W = \frac{m-n}{\left(Ae^{t(m-n)}\right)^{-1} + k}$$

$$W = \frac{m-n}{Le^{-t(m-n)} + k} \qquad \left(L = A^{-1}\right)$$

when $t = 0$, $W = W_i$, therefore:

$$W_i = \frac{m-n}{Le^0 + k}$$

$$= \frac{m-n}{L+k}$$

$$LW_i + kW_i = m-n$$

$$\therefore \ L = \frac{m-n-kW_i}{W_i}$$

Substituting L:

$$\boxed{W = \frac{(m-n)W_i}{(m-n-kW_i)e^{-(m-n)t} + kW_i}}$$

We can test this general equation by substituting $m = 0.1$, $n = 0.06$, $k = 0.00005$ and $W_i = 200$:

$$W = \frac{200(0.04)}{(0.04 - 0.01)e^{-0.04t} + 0.01}$$

$$= \frac{8}{0.03e^{-0.04t} + 0.01}$$

$$= \frac{8}{0.03e^{-0.04t} + 0.01} \times \frac{100}{100}$$

$$= \frac{800}{3e^{-0.04t} + 1}$$

This expression is identical to the one derived without the general formula, proving its credibility.

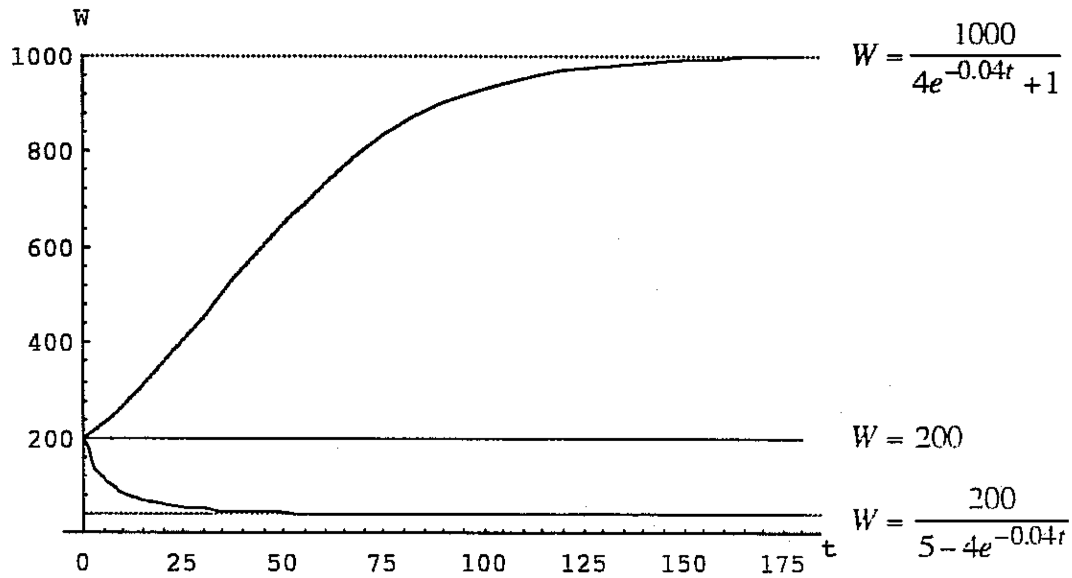
We can now explore k's effect by keeping m and n constant and varying k: Similarly to Model One there are three scenarios which could occur (and should be examined): $k \times W_i < m - n$, $k \times W_i = m - n$ and $k \times W_i > m - n$. Keeping m and n constant at 0.1 and 0.06 respectively (and $\therefore m - n = 0.04$) and choosing k values which illustrate the above scenarios produces the following equations:

m = 0.1, n = 0.06 and W_i = 200

	k	Equation
$200k < m - n$	0.00004	$W = \frac{8}{(0.04 - 200k)e^{-0.04t} + 200k}$ $= \frac{8}{(0.04 - 0.008)e^{-0.04t} + 0.008}$ $= \frac{8}{0.032e^{-0.04t} + 0.008}$ $W = \frac{8000}{32e^{-0.04t} + 8}$ $= \frac{1000}{4e^{-0.04t} + 1}$
$200k = m - n$	0.0002	$W = \frac{8}{(0.04 - 200k)e^{-0.04t} + 200k}$ $= \frac{8}{(0.04 - 0.04)e^{-0.04t} + 0.04}$ $= \frac{8}{0.04}$ $W = 200$

$200k > m-n$	0.001	$W=\dfrac{8}{(0.04-200k)e^{-0.04t}+200k}$ $=-\dfrac{8}{0.16e^{-0.04t}+0.2}$ $W=\dfrac{800}{20-16e^{-0.04t}}$ $=\dfrac{200}{5-4e^{-0.04t}}$

When all three are plotted:



It is seen that when $200k < m-n$ the population grows and approaches an asymptote. This would mean that the food supply hinders growth, but there are more wombats born to compensate for those dying of hunger. Mathematically, the reason for growth is because $W_i \times k < m-n$ resulting is positive co-efficient of e. We can also deduce that the ceiling limit of the population (ie. asymptote) is proportional to the food supply, ie. k. So if $W=\dfrac{W_i(m-n)}{(m-n-W_ik)e^{-t(m-n)}+k\times W_i}$ then as $t\to\infty$, $(m-n-W_ik)e^{-(m-n)t}\to 0$, meaning:

$$W_{\max}=\frac{W_i\times(m-n)}{W_i\times k}$$
$$=\frac{m-n}{k}$$

It should be noted that the asymptote is not affected by the initial population (ie. W_i) and that it is double the population at which maximum growth occurs ($\dfrac{m-n}{2k}$).

It is obvious that when $200k = m-n$ the population remains constant- there is no growth or decay. This would represent the situation where the number of wombats killed (from hunger or other reasons) equals the number being born. The mathematical reason is that $W_i \times k = m-n$, resulting in a co-efficent of 0 for e, reducing the population expression to $W = W_i$. We can mathematically determine for what values of k this will occur by finding when the ceiling limit equals W_i:

$$W_{max} = \frac{m-n}{k}$$

$$W_i = \frac{m-n}{k}$$

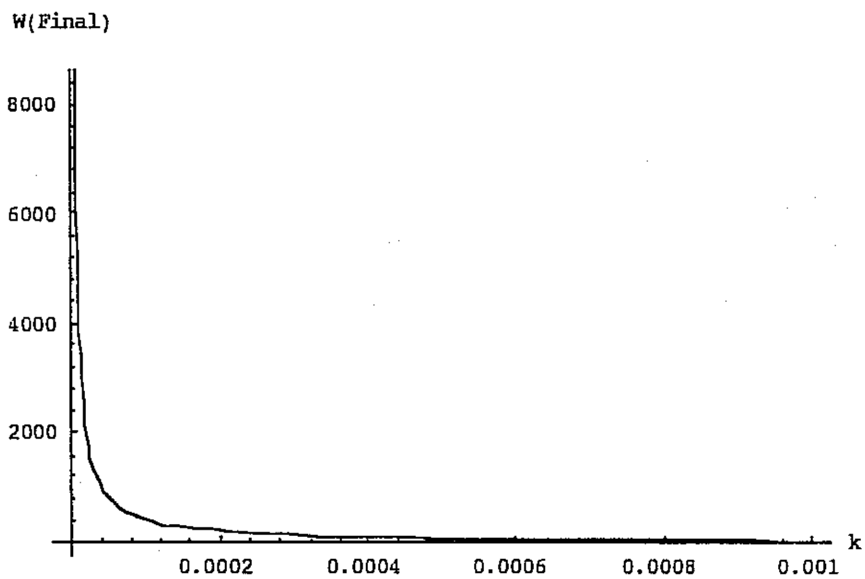
$$k = \frac{m-n}{W_i}$$

$W_i = 200$, so:

$$k_{\text{no growth}} = \frac{m-n}{200}$$

When $k > \dfrac{m-n}{W_i}$ we see that there is a decline in the curve until it steadily approcahes 0 population. This would represent the scenario in which there is not enough food to sustain even 200 wombats (ie. the number dying of hunger is greater than the number born). Mathematically, the reason for the decay is that $W_i \times k > m - n$, giving e and negative co-efficient.

k's effect on the asymptote (W_{final}) can be summarised by the following graph:



This clearly shows that $W_{final} \to 0$ as $k \to \infty$ and $W_{final} \to \infty$ as $k \to 0$.

In general, we can state that, as long as $m - n > 0$, when $k < \dfrac{m-n}{W_i}$ there will be growth in the wombat population, when $k = \dfrac{m-n}{W_i}$ then there will be no growth or decay, and when $k > \dfrac{m-n}{W_i}$ there will be a decay in the population. The maximum or minimum values of population (for growth and decay) is given by $W_{\text{max or min}} = \dfrac{m-n}{k}$.

Limitations

While Model Two proved to be better, k has a negative impact from $t = 0$) regardless of the number of wombats. In reality this would not be true for there would be a period of unrestricted growth after which food shortages became problematic. A better model would also having discrete data/points.

Conclusion

It was obvious that Model One, with its unrestricted exponential growth was an unrealistic model. While Model Two was much better, setting a limit to the number of sustainable wombats, it is still only a very rough guide. Further models which which consider as many factors as possible (such as weather) would vastly improve the predictions' accuracy.

Appendix A

If $m = 0.1$ and $n = 0.06$:

$$\frac{dW}{dt} = (0.1 - 0.06)W$$
$$= 0.04W$$

Inverting both sides of the equation:

$$\therefore \frac{dt}{dW} = \frac{1}{0.04W} = \frac{25}{W}$$

$$t = \int \frac{25}{W}\, dW$$
$$= 25 \times \int \frac{1}{W}\, dW$$
$$= 25 \log_e W + c$$

Transposing to make W the subject:

$$t - c = \frac{\log_e W}{0.04}$$
$$0.04(t - c) = \log_e W$$
$$W = e^{0.04(t-c)}$$
$$W = e^{0.04t} \times e^{-0.04c}$$
$$W = Ae^{0.04t} \qquad \left(\text{where the constant } A = e^{-0.04c}\right)$$

Evaluating A:
When $t = 0$, $W = 200$

$$\therefore 200 = Ae^{0.04(0)} = A$$

So $\boxed{W = 200e^{0.04t}}$

If we substitute the same values for m and n into our general equation:

$$W = 200e^{(m-n)t}$$
$$= 200e^{(0.1-0.06)t}$$
$$= 200e^{0.04t}$$

It can be seen that the two expressions are identical.

Acknowledgements

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References

Software:

- Wolfram Research Inc, *Mathematica: Student Version*, Champaign, 1993.