TITLE PAGE

# Problem 1 - The Great Escape

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# MAIN TEXT

## INTRODUCTION:

This problem investigates the relationships between flow rate of water, time, heights and volumes of water, and the size of a concrete block - upon which Matthew stands and from which he must escape whilst in a flask.

## VARIABLES:

- $h$ = height of water (metres) above the floor.
- $V(h)$ = volume of water (litres) in the empty flask for a given 'h'.
- $C$ = height of the concrete block (metres) above the stage. $\{C: 0 \leq C \leq 1.2\}$
- $C_{cm}$ = height of the concrete block (centimetres) above the stage. $\{C_{cm}: 0 \leq C_{cm} \leq 120\}$
- $\frac{dv}{dt}$ = water flow rate into flask (L / min).
- $V(t)$ = volume of water (litres) in the flask at 't' minutes $\{t: 0 \leq t \leq 10\}$ for a given $\frac{dv}{dt}$.
- $h(t)$ = height of water (metres) in the flask above the floor at 't' minutes.
- $h_s(t)$ = height of water (metres) in the flask above the stage at 't' minutes.
- $s$ = submersion time in minutes. $\{s: 0 \leq s \leq 2.2\}$
- $V_b$ = volume (litres) of concrete block.
- $V_m$ = volume (litres) of Matthew.
- $H$ = height (centimetres) of block above the stage.
- $H_m$ = height of Matthew.
- $r$ = radius (centimetres) of concrete block.
- $V_{req}$ = volume (litres) of water required to reach the top of Matthew's head when $V_m$ and $V_b$ are considered.

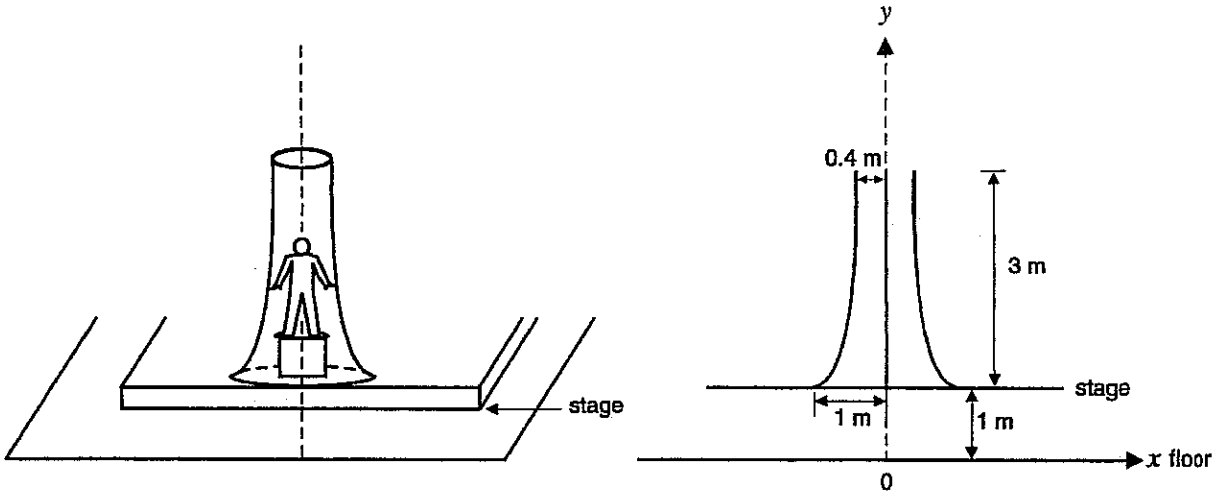
## INITIAL ASSUMPTIONS:

In the first four questions, it was assumed that both the block and Matthew were of negligible volume. In the final two questions, it was assumed that Matthew was a cylinder of fixed volume with radius (13 cm) and height (180 cm), and that the flask was built around the concrete block –allowing the block to have a radius greater than 40 cm (the smallest aperture of the flask). Throughout the problem, it was assumed that the water was rising from the bottom of the flask.

*For consistency, all numerical answers for volumes were given to the nearest litre – sufficient accuracy for a real-life application – whilst all heights were measured to the nearest millimetre.*

**Q1)** The longitudinal cross section takes the shape of the curve with equation $y = \frac{a}{x^2} + b$:

*Figure 1 – Flask Design*




Since there are two parameters, two co-ordinates are required to stipulate the equation:

- Two co-ordinates can be identified: (-0.4,4) and (1,1).

$\Rightarrow y(1) = a + b = 1$ **..... (1)**

$\Rightarrow y(-0.4) = \frac{a}{(-0.4)^2} + b = 4$

$\therefore 6.25a + b = 4$ **..... (2)**

From **(1)**, $b = 1 - a$

Substituting 'b' into **(2):** $6.25a + 1 - a = 4$

$\therefore 5.25a = 3$

$\therefore a = \frac{4}{7}$

$\therefore b = \frac{3}{7}$

Hence: $y = \frac{4}{7x^2} + \frac{3}{7}$ describes the cross-sectional shape.

**Q2)** Let h = height (metres) above the floor where $h \in [1,4]$

$\therefore$ The volume in the tank as a function of height above the floor [i.e. V(h)] is determined by rotating the function, $y = \frac{4}{7x^2} + \frac{3}{7}$, about the y-axis (i.e. a solid of revolution):

$V(h) = 1000\pi \int_1^h x^2 dy$ litres (since $1m^3 = 1000$ L)

Now: $y = \frac{4}{7x^2} + \frac{3}{7}$

$\therefore \frac{4}{7x^2} = y - \frac{3}{7}$

$\therefore \frac{4}{x^2} = 7y - 3$

$$\therefore x^2 = \frac{4}{7y-3}$$

Substituting '$x^2$': $V(h) = 1000\pi \int_1^h \frac{4}{7y-3}dy$ litres

$$= \frac{4000\pi}{7}\int_1^h \frac{7}{7y-3}dy$$

$$= \frac{4000\pi}{7}\left[\ln(7y-3)\right]_1^h$$

$$= \frac{4000\pi}{7}[\ln(7h-3) - \ln 4]$$

$$= \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right) \text{ litres.}$$

$$\therefore V(h) = \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right) \text{ litres.}$$

The total volume of water required to completely fill the flask occurs when h = 4:

$$\Rightarrow V(4) = \frac{4000\pi}{7}\ln\left(\frac{7(4)-3}{4}\right)$$

$$= \frac{4000\pi}{7}\ln\frac{25}{4}$$

$$\approx 3290 \text{ L}$$

**(a)** 3290 L

**(b)** $V(h) = \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right)$ L

**Q3)** Matthew's height = 180 cm = 1.8 m
Height of the concrete block = C m
∴ Distance from floor to the top of Matthew's head = 1.8 + C + 1 = (2.8 + C) m
Since Matthew plans to make his escape as the level of water reaches the top of his head, and it takes him 10 minutes to escape, then the height of water (from the floor) at 10 minutes must be (2.8 + C) m. (i.e. h = (2.8 + C)).

Flow rate of water = $\frac{dv}{dt}$ = 300 L / min

∴ Volume of water in the flask at 10 min = $10\frac{dv}{dt}$ = 3000 L

$$\Rightarrow V(2.8 + C) = 3000$$

$$\Rightarrow \frac{4000\pi}{7}\ln\left(\frac{7(2.8+C)-3}{4}\right) = 3000$$

$$\therefore \ln\left(\frac{7(2.8+C)-3}{4}\right) = \frac{21}{4\pi}$$

$$\therefore \frac{7(2.8+C)-3}{4} = e^{\frac{21}{4\pi}}$$

$$\therefore 2.8 + C = \frac{1}{7}(4e^{\frac{21}{4\pi}} + 3)$$

$$\therefore C = \frac{1}{7}(4e^{\frac{21}{4\pi}} + 3) - 2.8$$

$$\approx 0.6675 \text{ m}$$

$$\approx 66.8 \text{ cm}$$

$\Rightarrow$ if the concrete block is 66.8 cm high [$\frac{dv}{dt} = 300$ L / min], Matthew can escape at 10 minutes.

**Q4)** Flow Rate $= \frac{dv}{dt} = \beta$ L / min (where $\beta \in R$)

The exact volume of the flask was calculated to be $\frac{4000\pi}{7}\ln\left(\frac{25}{4}\right)$ L

The tank begins to overflow after 10 minutes:

$$\Rightarrow 10\,\frac{dv}{dt} = \frac{4000\pi}{7}\ln\left(\frac{25}{4}\right)$$

$$\therefore \frac{dv}{dt} = \frac{400\pi}{7}\ln\left(\frac{25}{4}\right) \text{ L / min } (\approx 329 \text{ L / min})$$

Hence, the rate at which water must be pumped into the flask so that it just begins to overflow after 10 minutes is 329 L / min.

Thus the exact volume of water in the flask at 't' minutes $= V(t) = \frac{dv}{dt}t = \frac{400\pi}{7}\ln\left(\frac{25}{4}\right)t$ L

N.B. $\{t: 0 \le t \le 10\}$

Therefore, the height (in metres) of water (from the floor) in the flask at time 't' can be determined:

$V(h) = V(t)$

$$\therefore \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right) = \frac{400\pi}{7}\ln\left(\frac{25}{4}\right)t$$

$$\therefore \ln\left(\frac{7h-3}{4}\right) = \frac{1}{10}\ln\left(\frac{25}{4}\right)t$$

$$\therefore \frac{7h-3}{4} = e^{\ln\left[\left(\frac{25}{4}\right)^{0.1t}\right]}$$

$$\therefore 7h - 3 = 4\left(\frac{25}{4}\right)^{0.1t}$$

$$\therefore h = \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{0.1t} + 3\right]$$

$$\therefore h(t) = \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{0.1t} + 3\right] \text{metres.}$$

But, the height (metres) of liquid in the flask (from the stage) [i.e. $h_s(t)$] at time 't' is $h(t) - 1$.

$$\Rightarrow h_s(t) = h(t) - 1$$

$$= \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{0.1t} + 3\right] - 1$$

$$= \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{0.1t} - 4\right]$$

$$= \frac{4}{7}\left[\left(\frac{25}{4}\right)^{0.1t} - 1\right]$$

$$\therefore h_s(t) = \frac{4}{7}\left[\left(\frac{25}{4}\right)^{0.1t} - 1\right] \text{metres.}$$

Matthew takes exactly 10 minutes to escape from his chains. Hence, at the instant that Matthew is free, the water begins to overflow. Therefore if he is completely submerged for a certain time 's' minutes, then the height of water must be at the top of his head at (10 – s) minutes [i.e. When t = (10 – s) min, h(t) = 2.8 + C]. Therefore, the height of the block as a function of submersion time can be determined:

$$\Rightarrow h(10\text{-}s) = 2.8 + C = \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{0.1(10-s)} + 3\right]$$

$$\therefore C + 2.8 = \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{1-0.1s} + 3\right]$$

$$\therefore C = \frac{1}{7}\left[4\left(\frac{25}{4}\right)^{1-0.1s} - 16.6\right] \text{ metres.}$$

$$\therefore C_{cm} = \frac{1}{7}\left[400\left(\frac{25}{4}\right)^{1-0.1s} - 1660\right] \text{ cm [i.e. } C_{cm} = 100C \text{ since 1 metre} = 100 \text{ cm]}$$

$$\therefore C_{cm}(s) = \frac{1}{7}\left[400\left(\frac{25}{4}\right)^{1-0.1s} - 1660\right] \text{cm}$$

Thus, if the submersion time is 1 minute, the height of the block required is $C_{cm}(1)$:

$$C_{cm}(1) = \frac{1}{7}\left[400\left(\frac{25}{4}\right)^{1-0.1} - 1660\right] \text{ cm.}$$

$$= \frac{1}{7}\left[400\left(\frac{25}{4}\right)^{0.9} - 1660\right] \text{ cm.}$$

$$\approx 60.2 \text{ cm.}$$

**Domain and range of $C_{cm}(s)$:**

Since $C_{cm} \geq 0$: $400\left(\frac{25}{4}\right)^{1-0.1s} - 1660 \geq 0$

$$\therefore \left(\frac{25}{4}\right)^{1-.01s} \geq \frac{83}{20}$$

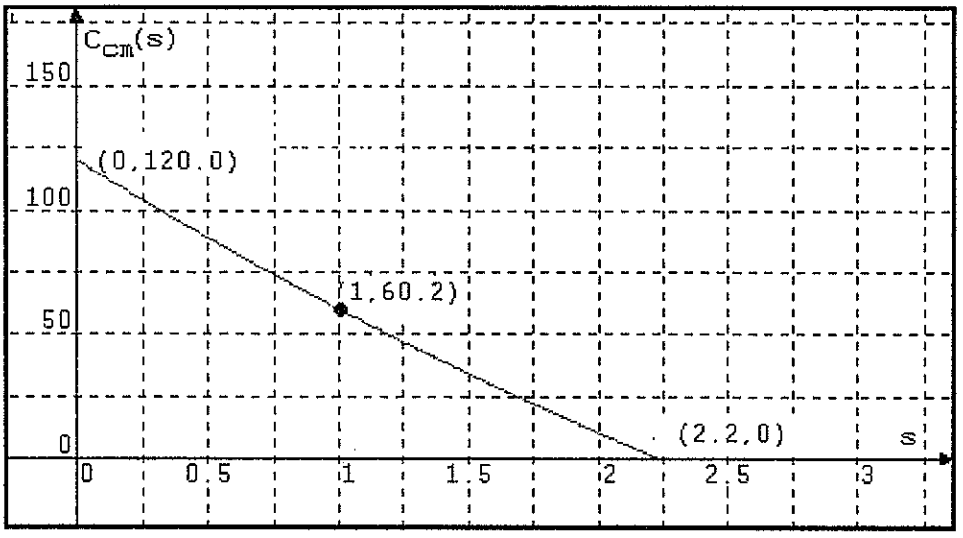
$$\therefore \quad 1\text{-}0.1s \geq \frac{\ln\frac{83}{20}}{\ln\frac{25}{4}}$$

$\therefore s \leq 10 - \dfrac{10\ln\frac{83}{20}}{\ln\frac{25}{4}}$

$\therefore s \leq 2.2$ minutes

Also $s \geq 0$:
Therefore, the domain of $C_{cm}(s)$ is: $\{s: 0 \leq s \leq 2.2\}$ – minutes.
It follows that the range of $C_{cm}(s)$ is: $\{C_{cm}: 0 \leq C_{cm} \leq 120.0\}$ – centimetres.

*Figure 2: Block Height vs Submersion Time:* $C_{cm}(s) = \dfrac{1}{7}\left[400\left(\dfrac{25}{4}\right)^{1-0.1s} - 1660\right]$




Because of physical constraints, the curve does not exhibit asymptotic behaviour (a usual characteristic of exponential patterns). Also, the approximately linear nature of $C_{cm}(s)$ is due to the small exponent, $(1 - 0.1s)$. As submersion time increases, the height of the block decreases – this is supported by a reduced volume of water being needed to be pumped in, to reach Matthew's head and so less time available for Matthew to breathe.
N.B. although submersion time is restricted by the height of the block (i.e. $C_{cm}$ is positive), 2 min and 14 sec, is also a very long time for Matthew to hold his breath (perhaps a limit before he experiences hypoxia).

**(a)** $\dfrac{dv}{dt} \approx 329$ L / min

**(b)** 60.2 cm

**(c)** $h(t) = \dfrac{1}{7}\left[4\left(\dfrac{25}{4}\right)^{0.1t} + 3\right]$ metres.

$h_s(t) = \dfrac{4}{7}\left[\left(\dfrac{25}{4}\right)^{0.1t} - 1\right]$ metres.

**(d) i.** $C_{cm}(s) = \frac{1}{7}\left[400\left(\frac{25}{4}\right)^{1-0.1s} - 1660\right]$ cm

** $\{s: 0 \leq s \leq 2.2\}$ - minutes
** $\{C_{cm}: 0 \leq C_{cm} \leq 120\}$ - centimetres

**ii.** The graph of $C_{cm}$ vs 's' – page 8.

**Q5)** Both the volumes of the block and Matthew reduce the volume of water required to be pumped in, to reach a certain height - considerations are therefore needed to account for this.
Volume of the cylindrical block ($V_b$) with radius ($r_b$) 13 cm, and height ($h_b$) H cm.
$\therefore V_b = \pi r_b^2 H = \pi(13)^2 H = 169\pi H \text{ cm}^3$
$\therefore V_b = 0.169\pi H$ litres [since 1000 $\text{cm}^3 = 1$ L]

Volume of Matthew ($V_m$): Matthew is assumed to be a cylinder with radius ($r_m$) 13 cm and height ($h_m$)180 cm.
$\therefore V_m = \pi r_m^2 h_m = \pi(13)^2(180) = 30{,}420\pi \text{ cm}^3$
$\therefore V_m = 30.42\pi$ litres (since 1000 $\text{cm}^3 = 1$ L)
Hence: $V_b + V_m = (0.169H + 30.42)\pi$ L – *total volume occupied by Matthew and the block.*

Height of Block = H cm = 0.01H m
Height of Matthew ($H_m$) = 1.8 m
Height of Stage = 1 m
$\therefore$ the distance from Matthew's head to the floor is: $H_m + 0.01H + 1 = (2.8 + 0.01H)$ m.
i.e. $h = 2.8 + 0.01H$

The volume of water required ($V_{req}$) to be pumped in, to just cover Matthew's head is the difference between the volume of water in the empty flask at (2.8 + 0.01H) metres from the floor and the total volume occupied by the block and Matthew:

Now: $V(h) = \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right)$

$$\Rightarrow V(2.8 + 0.01H) = \frac{4000\pi}{7}\ln\left(\frac{7(2.8+0.01H)-3}{4}\right)$$
$$= \frac{4000\pi}{7}\ln\left(\frac{16.6+0.07H}{4}\right)$$
$$= \frac{4000\pi}{7}\ln\left(\frac{1660+7H}{400}\right)$$

$$\Rightarrow V_{req} = V(2.8 + 0.01H) - (V_b + V_m)$$
$$= \frac{4000\pi}{7}\ln\left(\frac{1660+7H}{400}\right) - (0.169H + 30.42)\pi$$
$$= \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - 0.169H - 30.42\right] \text{ L}$$

$$\therefore V_{req}(H) = \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - 0.169H - 30.42\right] \text{ L} \quad [r_b = 13 \text{ cm}]$$

Therefore, if the block has a height of 60 cm and a radius of 13 cm, the volume of water needed to be pumped in, so that Matthew's head is just covered when standing upright is $V_{req}(60)$:

$$\Rightarrow V_{req}(60) = \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7(60)}{400}\right) - 0.169(60) - 30.42\right] \text{ L}$$

$$= \pi\left[\frac{4000}{7}\ln(5.2) - 40.56\right] \text{ L}$$

$$\approx 2832 \text{ L}$$

*Figure 3* – $V_{req}(H) = \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - 0.169H - 30.42\right]$ L

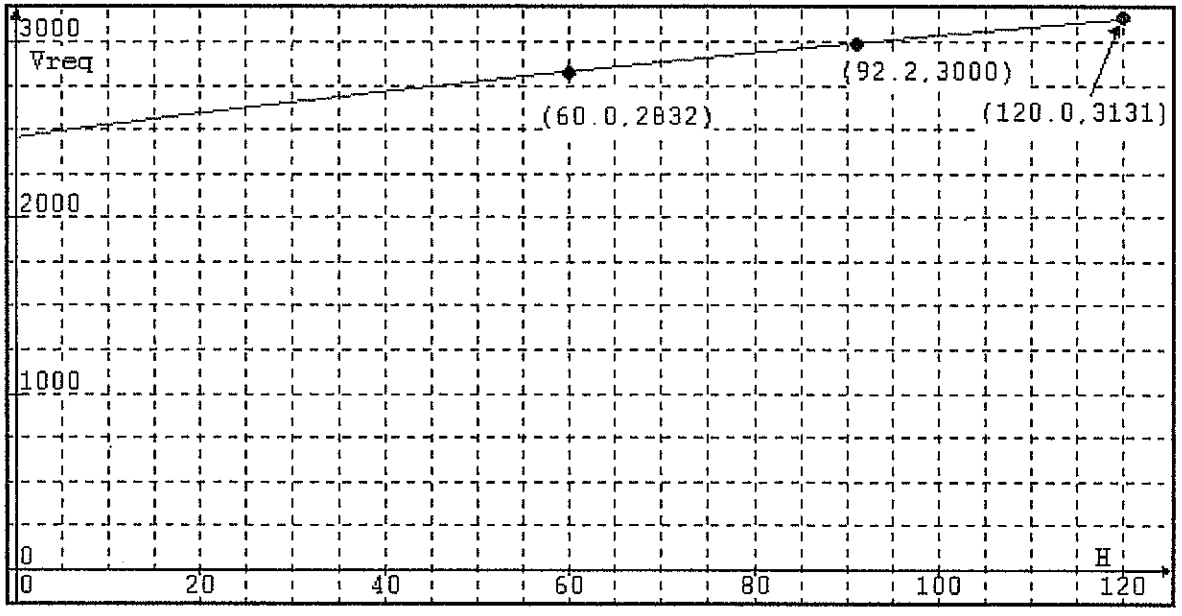



Figure 3 suggests that the extra height of water required to reach Matthew's head (due to a taller block) increases the required volume of water more than the increased volume of the block reduces it. Therefore, as H approaches 120.0 cm, $V_{req}(H)$ approaches 3131 L.

Now, given that $\frac{dv}{dt} = 300$ L / min, the volume of water pumped in, after ten minutes is $10\frac{dv}{dt}$:

$\Rightarrow$ Volume = 10 x 300 = 3000 L

*Comparing earlier when only 2832 L were required: since a larger volume of water is pumped in, the water level will be higher than earlier – hence, it is expected that H > 60 cm.*

$\therefore$ if Matthew's head is covered in exactly 10 minutes, $V_{req}(H) = 3000$:

$$\Rightarrow \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - 0.169H - 30.42\right] = 3000$$

$$\therefore 4000\ln\left(\frac{1660+7H}{400}\right) - 1.183H - 212.94 = \frac{21{,}000}{\pi}$$

$$\therefore 4000\ln(1660+7H) - 4000\ln 400 - 1.183H - 212.94 - \frac{21{,}000}{\pi} = 0$$

*Refer to appendix 1 for the spreadsheet solution of this equation.*

$\therefore$ when $H \approx 92.2$ cm, Matthew's head is covered in 10 minutes $[\frac{dv}{dt} = 300 \text{ L / min}]$.

**(a)** 2823 L

**(b) i.** $V_{req}(H) = \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - 0.169H - 30.42\right]$ L $\quad [r_b = 13 \text{ cm}]$

**ii.** 92.2 cm

**Q6)** Now consider a block, radius, r cm and height, H cm.

$V_b = \frac{\pi r^2 H}{1000}$ L (refer pg 9)

$V_m = 30.42\pi$ L (refer pg 9)

$\therefore V_b + V_m = \pi\left[\frac{r^2H}{1000} + 30.42\right]$ L - *total volume occupied by Matthew and the block.*

Distance from Matthew's head to the floor = (2.8 + 0.01H) m (refer pg 9)

Now $V(h) = \frac{4000\pi}{7}\ln\left(\frac{7h-3}{4}\right) \Rightarrow V(2.8 + 0.01H) = \frac{4000\pi}{7}\ln\left(\frac{16.6+0.07H}{4}\right)$ L.

$\Rightarrow$ the volume of water to be pumped in, to just cover Matthew's head $[V_{req}]$ is:

$V_{req} = V(2.8 + 0.01H) - (V_b + V_m)$

$= \frac{4000\pi}{7}\ln\left(\frac{16.6+0.07H}{4}\right) - \pi\left[\frac{r^2H}{1000} + 30.42\right]$

$= \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - \frac{r^2H}{1000} - 30.42\right]$ L

Hence, to achieve $V_{req}$ in exactly 10 minutes, flow rate $= \frac{V_{req}}{10}$

$\therefore \frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - \frac{r^2H}{1000} - 30.42\right]$ L / min.

**H and r:**

The maximum 'r' that allows H = 120 cm:

$y = \frac{4}{7x^2} + \frac{3}{7}$

$\therefore 1.2 + 1 = \frac{4}{7x^2} + \frac{3}{7}$

$\therefore x^2 = \frac{10}{31}$

$\therefore x = \sqrt{\frac{10}{31}}$ m

$= 100\sqrt{\frac{10}{31}}$ cm $(\approx 56.8$ cm$)$

Since 100x (cm) = r (cm) and it is assumed that $r \geq 13$ cm for a sturdy support, it follows:

- If $13 \leq r \leq 56.8$ then $0 < H \leq 120.0$ (cm)

But if r > 56.8 cm, the flask physically restricts H thus:
For $r > 56.8$, $H \leq \alpha$

$$\alpha: \quad \alpha = 100\left[\frac{4}{7(0.01r)^2} + \frac{3}{7} - 1\right] \text{ cm}$$

$$= \frac{400}{7}\left[\frac{10000}{r^2} - 1\right] \text{ cm.}$$

Hence:

- If $56.8 < r < 100$ then $0 < H < \frac{400}{7}\left[\frac{10000}{r^2} - 1\right]$

N.B. when r = 100 cm, H = 0 and thus the block has no real height (hence r < 100). Also, $H \leq 120$ cm, otherwise Matthew's head would be out of the water.

In practice however, a reasonable gap between the block and flask is needed to avoid chaotic water movement and pressure build-up due to the serious restriction to rising water. It is difficult to substantiate quantifying any adjustments and hence this qualitative analysis is all that is required.

**Flow Rate:**

Since an increased height of water in the flask increases $V_{req}$ more than a smaller block increases it, $V_{req}$ is a maximum when r = 13 cm and H = 120 cm. Hence $\frac{dv}{dt}$ is a maximum when r = 13 cm and H = 120 cm.

i.e. maximum $\frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660 + 7(120)}{400}\right) - \frac{(13)^2(120)}{1000} - 30.42\right]$

$$\approx 313 \text{ L / min.}$$

Now, for a given height, $\frac{dv}{dt}$ is smallest when 'r' is a maximum since $V_{req}$ is minimized for that height.

$\therefore$ since $y = \frac{4}{7x^2} + \frac{3}{7}$

$\Rightarrow$ Maximum 'r' for a specified H is:

$$0.01H + 1 = \frac{4}{7(0.01r)^2} + \frac{3}{7}$$

$$\therefore 0.07H + 7 = \frac{4}{7(0.01r)^2} + 3$$

$$\therefore \frac{(0.01r)^2}{4} = \frac{1}{0.07H + 4}$$

$$\therefore r^2 = \frac{40000}{0.07H + 4}$$

Substituting '$r^2$' into $V_{req}$ yields the minimum $V_{req}$ for H:

$$V_{req} = \pi\left[\frac{4000}{7}\ln\left(\frac{1660 + 7H}{400}\right) - \frac{40000}{0.07H + 4} \times \frac{H}{1000} - 30.42\right]$$

$$= \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - \frac{40H}{0.07H+4} - 30.42\right]$$

$$= \frac{4000\pi}{7}\ln\left(\frac{1660+7H}{400}\right) - \frac{40\pi H}{0.07H+4} - 30.42\pi$$

The local minimum for $V_{req}$ occurs when $\frac{dv_{req}}{dH} = 0$

$$\therefore \frac{dv_{req}}{dH} = \frac{4000\pi}{7} x \left(\frac{\frac{7}{400} x 400}{1660+7H}\right) - [40\pi(0.07H+4)^{-1} - 2.8\pi H(0.07H+4)^{-2}]$$

$$= \frac{4000\pi}{1660+7H} - \frac{160\pi}{(0.07H+4)^2}$$

$$\therefore \frac{dv_{req}}{dH} = 0 \text{ when } \frac{4000\pi}{1660+7H} = \frac{160\pi}{(0.07H+4)^2}$$

$$\therefore 4000\pi(0.0049H^2 + 16 + 0.56H) = 160\pi(1660+7H)$$

$$\therefore 0.1225H^2 + 14H + 400 = 1660 + 7H$$

$$\therefore 0.1225H^2 + 7H - 1260 = 0$$

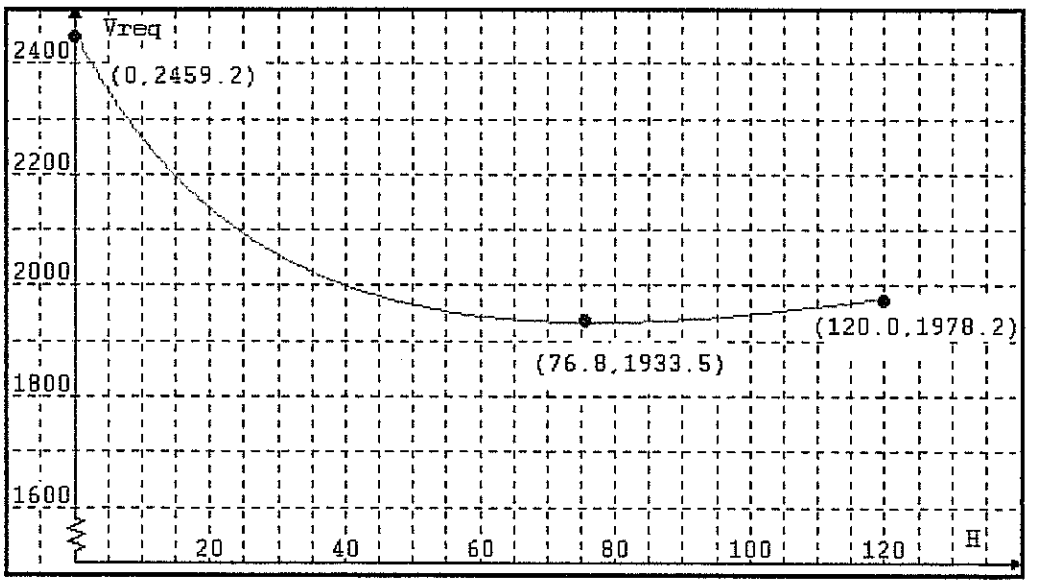
$$\therefore \text{since } H > 0,\ H = \frac{-7 + \sqrt{7^2 + (4x0.1225x1260)}}{2x0.1225}$$

$$\approx 76.8 \text{ cm}$$

$$\Rightarrow r = \sqrt{\frac{40{,}000}{0.07(76.8)+4}} \approx 65.3 \text{ cm}$$

*Figure 4 – Minimum $V_{req}$ for a specific H*

$$V_{req} = \pi\left[\frac{4000}{7}\ln\left(\frac{1660+7H}{400}\right) - \frac{40H}{0.07H+4} - 30.42\right]$$



From Figure 4, when H = 76.8 cm and r = 65.3 cm, an absolute minimum for $V_{req}$ occurs. Thus it follows that when H = 76.8 cm and r = 65.3 cm, an absolute minimum for $\frac{dv}{dt}$ occurs.

$$\therefore \text{Minimum } \frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660+7(76.8)}{400}\right) - \frac{(65.3)^2(76.8)}{1000} - 30.42\right] \text{ L / min}$$

$$\approx 193.35$$

$$\approx 193 \text{ L / min} \quad \text{(nearest litre)}$$

Therefore, although the pump can produce $180 \le \frac{dv}{dt} \le 330$ (L / min), flow rate is further restricted to $193 \le \frac{dv}{dt} \le 313$ (L / min).

**Flow Rate, H and r – summary:**

**1)** Minimum $\frac{dv}{dt}$: r = 65.3 cm (corresponds to maximum 'r' for H = 76.8 cm)

H = 76.8 cm

$\frac{dv}{dt} \approx 193$ L / min

**2)** Maximum $\frac{dv}{dt}$: r = 13.0 cm

H = 120.0 cm

$\frac{dv}{dt} \approx 313$ L / min

**3)** Verification of Q5) (b) (ii): r = 13.0 cm

H = 92.2 cm

$$\Rightarrow \frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660+7(92.2)}{400}\right) - \frac{(13.0)^2(92.2)}{1000} - 30.42\right] \approx 300 \text{ L / min}$$

**QED**

4) When r = 13.0 cm
H = 0.1 cm

then $\frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660+7(0.1)}{400}\right) - \frac{(13.0)^2(0.1)}{1000} - 30.42\right] \approx 246 \text{ L / min}$

This flow rate is smaller than the maximum flow rate since the height of water required to reach Matthew's head is almost a minimum. This verifies that the height to which the water must rise influences $\frac{dv}{dt}$ and $V_{req}$ more than the volume of the block.

**5)** It can also be seen that the minimum $\frac{dv}{dt}$ does not occur when r is large and H is small:

r = 99.0 cm
H = 1.0 cm

$$\frac{dv}{dt} = \frac{\pi}{10}\left[\frac{4000}{7}\ln\left(\frac{1660+7(1.0)}{400}\right) - \frac{(99.0)^2(1.0)}{1000} - 30.42\right] \approx 244 \text{ L / min}$$

[i.e. $244 >$ minimum $\frac{dv}{dt}$]

Hence, the minimum $\frac{dv}{dt}$ occurs when a balance is struck between H and r (pg 14).

# APPENDICES

## APPENDIX ONE

**Finding the solution of the equation:**

$$4000\ln(1660 + 7H) - 4000\ln 400 - 1.183H - 212.94 - \frac{21{,}000}{\pi} = 0$$

| H | f(H) |
|---|---|
| 92 | -2.533710156 |
| 92.01 | -2.424014224 |
| 92.02 | -2.314321984 |
| 92.03 | -2.204633436 |
| 92.04 | -2.09494858 |
| 92.05 | -1.985267415 |
| 92.06 | -1.875589941 |
| 92.07 | -1.765916158 |
| 92.08 | -1.656246066 |
| 92.09 | -1.546579664 |
| 92.1 | -1.436916953 |
| 92.11 | -1.327257931 |
| 92.12 | -1.2176026 |
| 92.13 | -1.107950957 |
| 92.14 | -0.998303005 |
| 92.15 | -0.888658741 |
| 92.16 | -0.779018166 |
| 92.17 | -0.66938128 |
| 92.18 | -0.559748082 |
| 92.19 | -0.450118573 |
| 92.2 | -0.340492751 |
| 92.21 | -0.230870617 |
| 92.22 | -0.121252171 |
| 92.23 | -0.011637412 |
| 92.24 | 0.09797366 |
| 92.25 | 0.207581045 |
| 92.26 | 0.317184743 |
| 92.27 | 0.426784755 |
| 92.28 | 0.53638108 |
| 92.29 | 0.64597372 |
| 92.3 | 0.755562674 |
| 92.31 | 0.865147942 |
| 92.32 | 0.974729526 |
| 92.33 | 1.084307424 |

Equation placed into the spreadsheet for f(H) to perform the search:
= 4000*ln(1660 + 7*H) – 4000*ln400 – 1.183*H – 212.94 – 21,000/pi
**This spreadsheet is all that is required to specify H to the nearest millimetre (92.2 cm).**



tsfx