VCE Mathematics 1994
Common Assessment Task 1: Investigative Project

MATHEMATICAL METHODS
UNITS 3 AND 4

COVER SHEET

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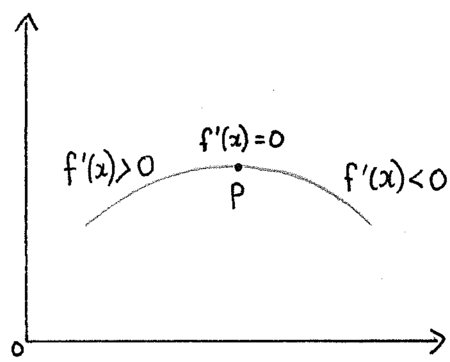
Introduction:

To begin this project, the given topic will be discussed in further detail to point out the direction taken, in order to answer the aims. This project on wrapping paper mainly involves the application of maximum - minimum theory. According to the aims, it was expected that the relationship between the volume, total surface area and dimensions of a solid were examined in an attempt to determine the conditions necessary for a solid, or its wrapping, to have a minimum surface area. Throughout the project, the volume (V) was kept fixed and the dimensions of the box x,y,z (x =length, y = width, z = height), varied at different stages.

The solids investigated in this project were the rectangular box and the cylinder. For both solids, an application of differentiation was required in order to obtain results and answer the given aims. An important part of the project was to be able to determine whether or not a turning point was a local maximum or a local minimum. The way of distinguishing between the two was to sketch graphs of the functions on a computer. If the graph was of following shape:

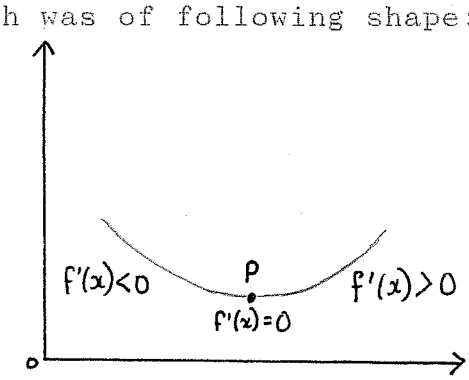
$x<a,\ f'(x)>0$
$x=a,\ f'(x)=0$
$x>a,\ f'(x)<0$



CONVEX DOWNWARDS

then the turning point would be a maximum as it is shaped convex downwards. If the graph was of following shape:

$x<a,\ f'(x)<0$
$x=a,\ f'(x)=0$
$x>a,\ f'(x)>0$



CONCAVE UPWARDS.

then the turning point would be a minimum as it is shaped concave upwards. However, this method was not used during the project. Alternatively, a quicker method was used to algebraically determine maxima and minima. This method was the second derivative. It enabled us to discriminate between maximum and minimum values by identifying maximum values as being negative and minimum values as being positive.

A limitation during the project was that the domain for all x values had to be as follows:

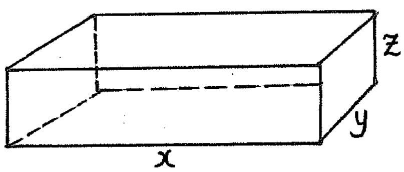
$$x \in R+$$

All values for x were positive as it was unrealistic to have negative lengths, widths and heights for the results.

RECTANGULAR BOX

In the following section, a rectangular box of fixed volume (V) and dimensions (x) - length, (y) - width, (z) - height, was investigated.

Firstly, it was required to show that the surface area of the box was a minimum when $x = y$. The volume (V) and the dimension (z) were fixed, while x and y varied.



Volume (V) = xyz
= length $\times$ width $\times$ height

Since (V) and (z) were fixed, xy was made to equal V/z (constant), by transposing the formula $V = xyz$ into $xy = V/z$ and hence letting $y = \frac{V}{z}x^{-1}$ so that the equation for surface area did not have more than one variable. This led to considering another limitation during the project. Calculus could only be used to differentiate if there was only one variable in the equation. Therefore, in order to differentiate, the equations had to be reduced to one variable.
The equation for surface area was as follows:

$$A = 2(xy + yz + zx)$$
$$A = 2[xy + z(x+y)]$$ - taking out a common factor of z.

In order to find the minimum surface area of the box, the minimum value of (x+y) had to found, since x and y were variables.
Substituting $y = \frac{Vx^{-1}}{z}$ into the surface area equation:

$$A = 2\left[\frac{V}{z} + z(x+y)\right]$$

Find the minimum value of (x+y)

Let $S = x+y$

$S = x + \frac{V}{z}x^{-1}$

Find the derivative $\frac{ds}{dx}$

$$\frac{ds}{dx} = 1 - \frac{V}{zx^2}$$

At the turning point $\frac{ds}{dx} = 0$

therefore, $1 - \frac{V}{zx^2} = 0$

$\frac{V}{zx^2} = 1$ (cross multiply both sides)

$$V = zx^2 \text{ (which is also equal to } xyz)$$
$$\cancel{z}x^2 = xy\cancel{z}$$
$$x = y \quad (1:1)$$

Therefore, a minimum surface area occurs when the box has a square base (ie $x = y$)
The shape of the box is a cube as $x = y = z$. To show that $x = y$ yields a minimum value for the surface area, the second derivative was found.

$$\frac{ds}{dx} = 1 - \frac{Vx^{-2}}{z}$$

Find the second derivative:

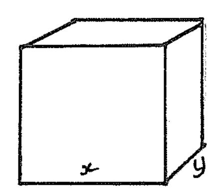
$$\frac{d^2s}{dx^2} = \frac{2Vx^{-3}}{z} > 0 \text{ for } x \in R^+$$

Since (V) and z (height) are both positive, then the turning point is a minimum.

DETERMINING THE SHAPE OF THE BOX

In this section, it was required to determine the shape of the box with the minimum surface area. The value of z varied, while $x = y$ (see appendix 1 for calculations).
It was found that the volume of the box was z^3, hence giving the end result as $x = z$. However, the shape of the box could have been determined without having to do any calculations. Since x, y, z are variables, this means that they are interchangeable due to the symmetry of the problem therefore, when $x = y$, we can also say $y = x$ and $z = x$ or $z = y$ and $y = z$. We can take this a step further by saying that $z = x = y$, $y = x = z$ and $x = y = z$, therefore resulting in the shape of a perfect cube, with dimensions x, y, z. The second derivative was applied and it was found that the surface area was a minimum.

Perfect Cube



TOP OF RECTANGULAR BOX IS REMOVED

In this case, the top of the rectangular box is removed so that it has five sides instead of six. This changed the original surface area formula into the following:

V is fixed
$$V = xyz$$
$$V = x^2 z$$
$$x^2 = \frac{V}{z}$$
$$x^2 = Vz^{-1}$$

$$A = xy + 2yz + 2xz$$
$\uparrow$ signifies one lid only

$$A = xy + 2z(x + y)$$ – take out common factor of z

Assuming that $x = y$,
$$A = x^2 + 2z(2x)$$ – substituting for x in order to reduce equation to one variable.
$$A = Vz^{-1} + 2z(2x)$$
$$A = Vz^{-1} + 4\sqrt{V}\, z^{-\frac{1}{2}}$$

Finding the derivative:

$$\frac{dA}{dz} = \frac{-V}{z^2} + \frac{2\sqrt{V}}{\sqrt{z}}$$

at turning point, $\frac{dA}{dz} = 0$

then $-\frac{V}{z^2} + \frac{2\sqrt{V}}{\sqrt{z}} = 0$

$\frac{2\sqrt{V}}{\sqrt{z}} = \frac{V}{z^2}$ (cross multiply) $\quad 2\sqrt{V}\,z^2 = V\sqrt{z}$

Squaring both sides to get rid of square root signs.

$4Vz^4 = V^2 z$ ($\div$ both sides by V)

$4z^4 = Vz$ ($\div$ both sides by z)

$4z^3 = V$ (which is also equal to $x^2 z$)

$z^3 = \frac{V}{4}$

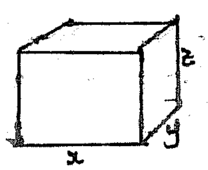
$4z^3 = x^2 z$ ($\div$ by z)

$4z^2 = x^2$ (take $\sqrt{\ }$ of both sides)

$2z = x$

$z = \frac{x}{2}$ (2:2:1)

Therefore, the box has a square base since x = y and the height (z) is half the length of x. It has the following shape:

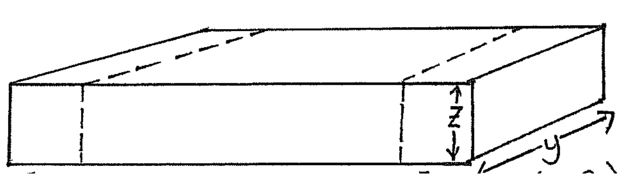


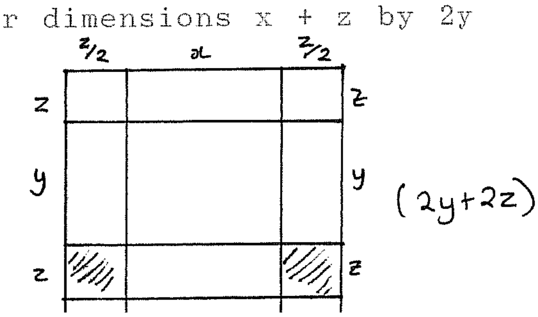
1/2 a cube (cuboid)

The second derivative was found and showed that the dimensions x, y, 2z yielded a minimum surface area, as all values were positive. (see appendix 2)

BOX IS WRAPPED IN A CONVENTIONAL WAY

The next step was to find the dimensions of the box for which the area of wrapping was a minimum. Previously, it was shown that a box with dimensions x × y × z had a minimum surface area when x = y. In this case, the box is wrapped with paper dimensions x + z by 2y + 2z as follows:





The minimum surface area of the paper was found by keeping z fixed and assuming that $x = y$ (see appendix 3 for calculations), therefore the box having a square base.

$$A = (x + z)\,2(y + z)$$

The volume, $V = xyz$

If $x = y$ then

$x = \dfrac{\sqrt{V}}{\sqrt{z}}$ which is also equal to y

Therefore $A = \left(\dfrac{\sqrt{V}}{\sqrt{z}} + z\right)\left(\dfrac{\sqrt{V}}{\sqrt{z}} + z\right)$

$$A = 2\left(\frac{V}{z} + 2\sqrt{V}z + z^2\right)$$

Thus $\dfrac{dA}{dz} = 2\left(-\dfrac{V}{z^2} + \dfrac{\sqrt{V}}{\sqrt{z}} + 2z\right)$

We find a turning point when $\dfrac{dA}{dz} = 0$

$$-\frac{V}{z^2} + \frac{\sqrt{V}}{\sqrt{z}} + 2z = 0$$

$$-V + \sqrt{V}\,z^{3/2} + 2z^3 = 0$$

Let $q = z^{3/2}$

$$-V + \sqrt{V}\,q + 2q^2 = 0$$

We can now solve for q by using the quadratic formula.

$$q = \frac{-\sqrt{V} \mp \sqrt{V - 4 \times 2 \times -V}}{2 \times 2}$$

$$= \frac{-\sqrt{V} \pm \sqrt{9V}}{4}$$

$$= \frac{-\sqrt{V} \pm 3\sqrt{V}}{4}$$

$$q = -\sqrt{V} \text{ or } \frac{\sqrt{V}}{2}$$

Since $q = -\sqrt{V}$ is impossible (ie cannot have a negative volume), then

$$q = \frac{\sqrt{V}}{2} \quad \text{or} \quad z = \sqrt[3]{\frac{V}{4}}$$

The second derivative was found and showed that the dimensions yielded a minimum surface area (see appendix 4).

The dimensions of the box were therfore:

$$z^3 = \frac{V}{4} \qquad z^3 = \frac{x^2 z}{4}$$

$$z^2 = \frac{x^2}{4}$$

$$z = \frac{x}{2}$$

$A = 2(x + z)(y + z)$ - substitute $z = \dfrac{x}{2}$

$$2 \times \frac{3x}{2} \times \frac{3x}{2} = \frac{9x}{2}$$

$x = y$ therefore $\dfrac{9x}{2} = \dfrac{9y}{2}$

and $z = \dfrac{x}{2}$ therefore $= 9z$.

$$\frac{9x}{2}, \frac{9y}{2}, 9z \quad (2:2:1)$$

Therefore, the dimension z, is half the length of the dimensions x and y. (2:2:1)

RECTANGULAR BOX: OVERLAPPING

The next step was to investigate whether or not the dimensions of the box would change if the paper overlapped itself in some way. Firstly, it was decided that the side lengths would be multiplied by b ($b = k_1(x + z)$ ie b is some proportional increase in the length) and a ($a = 2k_2(y + z)$ ie a is some proportional increase in the width.

Therefore, the dimensions were:

$$x + z + k_1(x + z) \text{ by } 2(y + z) + 2k_2(y + z)$$

or

$$(x + z)(1 + k_1) \text{ by } 2(y+z)(1 + k_2)$$

The area of the paper is now:

$$A = 2(x + z)(y + z)(1 + k_1)(1 + k_2)$$

To minimise the area, a substitution was made for x and y.

$$A = 2\left(\frac{\sqrt{V}}{\sqrt{z}} + z\right)\left(\frac{\sqrt{V}}{\sqrt{z}} + z\right)(1 + k_1)(1 + k_2)$$

$$\frac{dA}{dz} = 2\left(-\frac{V}{z^2} + \frac{\sqrt{V}}{\sqrt{z}} + 2z\right)(1 + k_1)(1 + k_2)$$

Therefore, if we let $\frac{dA}{dz} = 0$,

we obtain the same results as for when the box was wrapped up without any overlap.

Therefore $z = \sqrt[3]{\frac{V}{4}} = \frac{x}{2}$

Next, the wrapping paper was overlapped in a different way. It was decided that an amount of $\frac{1}{2}y$ and $\frac{1}{2}z$ would be added to

each side length of the box. In order to make the overlap realistic, the amount of overlap was limited to a fractional value between (0 - 1). The smallest value which could have been chosen was 0, however this meant that there would be no overlap. The largest value which could have been chosen was 1, but as people do not usually overlap by very large amounts, this value would have been unrealistic.

It was found that the dimensions of the box changed when the above amounts were added to the side lengths of the box (see appendix 5 for calculations).

The dimensions were as follows:

$$x \text{ by } \frac{8}{15}x \text{ by } \frac{x}{3}$$

Ratio (15:8:5)

This shows that the dimensions of the box have varied. The ratio (15:8:5), indicates that x has the longest side length, while y is nearly half the length of x and z is 1/3 the length of x.

To investigate further, it was decided to overlap the wrapping paper by:

1) adding an amount of 2k to each side length of the box (see

appendix 6 for calculations). However, the equation obtained could not be solved due to the limitation of having only one variable in the equation. In this case, both x and y could not be replaced, therefore it was impossible to differentiate. The dimensions of the box were not found, however it was assumed that this method of overlap was the most realistic.

2) adding an amount of 2bz and ay to each side length of the box (see appendix 7 for calculations). This particular means of overlap involved a more general case. However, due to the difficulty of the equations, the dimensions of the box were not found. To reduce the difficulty, numbers could have been substituted in the equations. This may have resulted in obtaining the dimensions which assumingly would have been different to those obtained when the box was wrapped without any overlap.

EXTENSION: CYLINDER

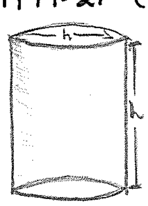
As an extension a cylinder was investigated as it was more realistic to wrap than a triangular prism.
To begin this section, a general formula was used to work out the dimensions of the cylinder when the lid was kept on and when the lid was removed.

$$A = 2\pi rh + n\pi r$$

n = 1 (cylinder without a lid)
n = 2 (cylinder with lid)
It was found that when n = 2, h = 2r and therefore $r = \frac{h}{2}$ (see appendix 8).

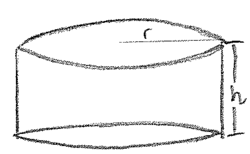
When n = 1, h = r and therefore r = h. The dimensions of the cylinder therefore varied when the lid was on and when the lid was removed.

When $h=2r$ $(n=2)$ $r=\frac{h}{2}$ $(2:1)$



The height and diameter are equal therefore, the square cross section through the axis of the cylinder yields a perfect square.

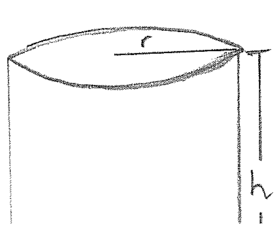
When $h=r$ $(n=1)$ $r=h$ $(1:1)$



The height and radius are equal therefore the square cross section through the axis of the cylinder yields a rectangle .

CYLINDER IS WRAPPED IN A CONVENTIONAL WAY

A cylinder with dimensions r, h was wrapped up so that the area was as follows:



$$V = \pi r^2 h$$
$$h = \frac{V}{\pi r^2}$$

$$A = 2\pi r(h + 2r)$$

Substitute $h = \dfrac{V}{\pi r^2}$ into the formula

$$A = 2\pi r\left(2r + \frac{V}{\pi r^2}\right)$$

$$A = 4\pi r^2 + \frac{2\pi r V}{\pi r}$$

$$A = 4\pi r^2 + 2Vr$$

Find the derivative

$$\frac{dA}{dr} = 8\pi r - \frac{2V}{r^2}$$

At turning point $\dfrac{dA}{dr} = 0$

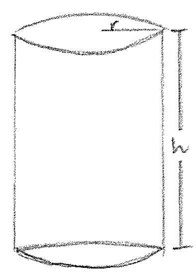
then $8\pi r = \dfrac{2V}{r^2}$ (cross multiply)

$$r^3 = \frac{2V}{48\pi}$$

$$V = 4\pi r^3 = \pi r^2 h$$

$$4\pi r = \pi r^2 h$$

$$h = 4r \qquad r = \frac{h}{4} \quad (4:1)$$



This indicated that the radius of the cylinder was a 1/4 length of the height

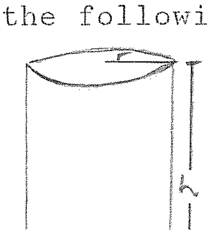
Second derivative was found and showed that the area was a minimum. (see appendix 9)

CYLINDER: OVERLAPPING

This section investigated whether or not an overlap in the wrapping paper would change the dimensions of the cylinder. Firstly, the dimensions were multiplied by k (see appendix 10 for calculations). However, as for the box, the dimensions of the cylinder stayed the same. Next, an amount of r and πr were added to the dimensions of the cylinder (see appendix 11). By adding the above amounts, the dimensions of the cylinder changed. The relationship between h and r became the following:

$$h = 6r$$

$$r = \frac{h}{6} \quad (6:1)$$



This indicated that the radius of the cylinder was a 1/6 length of the height.
During this section, it was observed that the amounts added to the dimensions of the cylinder were unrealistic. In order for the overlap to be realistic, the amounts added should have been limited to fractional values between (0 -1) - (as for the box). In this case, πr was added to the cylinder, meaning that the top of the cylinder was overlapped by a total of 3.142 times (value of π) This amount is quite unrealistic as people do not usually overlap by more than a couple of centimetres. Instead, $\frac{1}{2}r$ or $\frac{1}{4}r$ should have been added to the dimensions in order to make the overlap realistic.

CONCLUSION AND EVALUATION

On completion of this project, various relationships between the dimensions of both the rectangular box and the cylinder were established. The dimensions of the closed box changed when the lid was removed. This result also occurred for the cylinder. In both cases, the surface area of the solids had been reduced (altered), while the volume was kept fixed, therefore causing the dimensions of the solids to change. It seems reasonable to conclude that by altering the surface area of a solid, its dimensions will change, however definite conclusions cannot be derived from these results as only two solids were investigated.

When the rectangular box was wrapped in a conventional way, its dimensions were found to be as follows: 2:2:1. However when the wrapping paper was overlapped, the dimensions of the box changed. When overlap was chosen in a certain way (ie dimensions were multiplied by amounts a and b), the dimensions of the box did not change. However, when the following overlap was chosen (addition of $\frac{1}{2}z$ and $\frac{1}{2}y$ to the dimensions), it was found that the dimensions of the box did change (15:8:5). However, it would not be very realistic if a person overlapped by an exact amount of $\frac{1}{2}y$ and $\frac{1}{2}z$.
Realistically, people overlap by small amounts of 2-3 cm, no matter what the dimensions of the box are. The most realistic way of overlapping would have been to add a constant (k) to the paper, however the equation could not be solved due to the limitation of being unable to differentiate when there was more than one variable in the equation. The results obtained for the cylinder, were quite similar to those obtained for the box. The dimensions of the cylinder changed from 4:1 (with no overlap) to 6:1 when overlap was expressed in the following way (r and π r were added to the dimensions of the cylinder). It can therefore be stated that the dimensions of the box, cylinder and probably other solids will change when overlap is expressed as an addition to the dimensions. eg for the box:(2y + 2z + <u>1/2y</u>), for the cylinder: (2 r + <u>π r</u>). The mathematical techniques used throughout the project (calculus and algebra), were quite satisfactory as they assisted in obtaining the correct results and therefore allowed the given aims to be

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answered. In summary, my mathematical model has compromised real world applicability for solvability of the equations. One possible further extension could be to investigate other shapes by looking at the same principles as to those applied for the box and the cylinder. Also, an approximation technique could be used to look at a more realistic case (overlapping by 2-3 cm).

MATHEMATICAL TOOLS USED

Basic mathematical functions: addition, multiplication etc.
Calculus: Differentiation
Algebra : to solve equations
Volume formulae
ANU Graphs on the Apple Macintosh Computer

ACKNOWLEDGEMENTS

My teacher who taught me differentiation.
Fellow mathematical methods students for their support during the project.

REFERENCES

Rehill and McAuliffe. Mathematical methods Units 3 and 4 Melbourne; Macmillan Education Australia PTY LTD, 1993

Appendices.

Appendix ① $\quad$ V is fixed $\quad x = y \quad V = x^2 z$

$$x^2 = \frac{V}{z}$$

$$A = 2[Vz^{-1} + z(x+x)]$$

$$A = 2Vz^{-1} + 4\sqrt{V}\, z^{-\frac{1}{2}}$$

$$\frac{dA}{dx} = -\frac{2V}{z^2} - \frac{2\sqrt{V}}{\sqrt{z}}$$

at the turning point $\frac{dA}{dz} = 0$.

therefore $\quad -\frac{2V}{z^2} + \frac{2\sqrt{V}}{\sqrt{z}} = 0$

$$\frac{\sqrt{V}}{\sqrt{z}} = \frac{V}{z^2}$$

$$V\sqrt{z} = \sqrt{V} \times z^2$$

$$V^2 z = V \times z^4$$

$$V^2 = V \times z^3$$

$$V = z^3$$

$$z^3 = x^2 z$$

$$z^2 = x^2$$

$$x = z$$

$$\frac{d^2A}{dx^2} = \frac{4V}{z^3} + \sqrt{4}\, z^{3/2} \quad (\text{+ve} \therefore \text{ is a minimum}).$$

Appendix ②

$$\frac{dA}{dz} = -Vz^{-2} + 2\sqrt{V} z^{-\frac{1}{2}}$$

$$\frac{d^2A}{dz^2} = \frac{2V}{z^3} - \frac{\sqrt{V}}{z\sqrt{z}}$$

when $z^3 = \frac{V}{4}$

$$\frac{d^2A}{dz^2} = \frac{2V \times 4}{V} - \frac{2\sqrt{V}}{\ldots}\left(\frac{\times\sqrt{V}}{\ldots}\right) = 8 - 2 = 6$$

Appendix 3

$V = xyz \qquad z \text{ is fixed}$

$y = \frac{V}{xz}.$

$A = (2y + 2z)(x + z) \quad \left(\text{Sub } y = \frac{V}{xz}\right)$

$\left(\frac{2V}{xz} + 2z\right)(x + z)$

$\frac{2Vx}{xz} + \frac{2Vz}{xz} + 2zx + 2z^2$

$A(x) = 2Vz^{-1} + \frac{2V}{x} + 2zx + 2z^2$

$A'(x) = -2Vx^{-2} + 2z = 0$

$2z \times x^2 = \frac{2V}{\cancel{x^2}} \times \cancel{x^2}$

$x^2 = \frac{\cancel{2}V}{\cancel{2}z}$

$x = \sqrt{\frac{V}{z}}$

$y = \frac{V}{\sqrt{\frac{V}{z}}\, z}$

$y = \frac{V \times z^{\frac{1}{2}}}{V^{\frac{1}{2}}\, z}$

$y = \frac{V^{\frac{1}{2}}}{z^{\frac{1}{2}}}$

$y = \sqrt{\frac{V}{z}}$

Therefore $x = y$.

Appendix ④ To check we have a minimum.

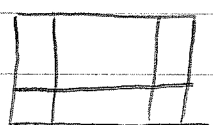
$$\frac{d^2A}{dz^2} = 2\left(\frac{2V}{z^3} - \frac{1}{2}\frac{\sqrt{V}}{z^{3/2}} + 2\right)$$

when $z = \sqrt[3]{\frac{V}{4}}$ ie $z = \frac{V}{4}$ we get

$$\frac{d^2A}{dx^2} = 2\left(\frac{2V}{V/4} - \frac{1}{2}\frac{\sqrt{V}}{\sqrt{V/4}} + 2\right)$$

$$= 16 - 1 + 2$$
$$= 17 \quad > 0 \therefore \text{ minimum.}$$

Appendix ⑤.



$(2x + 2y + \frac{1}{2}y)$

$\frac{1}{4}z$ $\frac{1}{4}z$

$(x + z + \frac{1}{2}z)$

$V = xyz$ (V) and (z) are fixed.

$y = \frac{V}{xz}$.

$$A(x) = (x + z + \tfrac{1}{2}z)(2y + 2z + \tfrac{1}{2}y)$$
$$= (x + \tfrac{3}{2}z)(2z + \tfrac{5}{2}y)$$
$$= (2zx + \tfrac{5}{2}xy + 3z^2 + \tfrac{15}{4}zy)$$
$$= 2zx + \frac{5}{2}\frac{\cancel{x}V}{\cancel{x}z} + 3z^2 + \frac{15}{4}\frac{\cancel{z}V}{xz} \quad \left(\text{Sub } y = \frac{V}{xz}\right)$$

$$A(x) = 2zx + \tfrac{5}{2}Vz^{-1} + 3z^2 + \tfrac{15}{4}Vx^{-1}$$

$$\frac{dA}{dx} = 2z - \frac{15}{4}Vx^{-2} = 0$$

$$2z = \frac{\frac{15}{4}V}{x^2}$$

$$x^2 = \frac{\frac{15}{4}V}{2z}$$

$$x = \sqrt{\frac{\frac{15}{4}V}{2z}} = \sqrt{\frac{15V}{8z}}$$

To find y $\Rightarrow$ $y = \dfrac{V}{xz}$ $\left(\text{sub } x = \sqrt{\dfrac{15V}{8z}}\right)$.

$$y = \frac{V}{\left(\frac{15V}{8z}\right)^{\frac{1}{2}} z}$$

$$y = \frac{V \times (8z)^{\frac{1}{2}}}{(15V)^{\frac{1}{2}} z}$$

$$y = \frac{V \times 8^{\frac{1}{2}} z^{\frac{1}{2}}}{15^{\frac{1}{2}} \times V^{\frac{1}{2}} z}$$

$$y = \frac{V^{\frac{1}{2}} \times 8^{\frac{1}{2}}}{z^{\frac{1}{2}} \times 15^{\frac{1}{2}}} = \sqrt{\frac{8V}{15z}}$$

— put in terms of x.

$$x = \sqrt{\frac{15}{8}} \times \frac{\sqrt{V}}{\sqrt{z}}$$

$$\sqrt{\frac{8}{15}}\, x = \sqrt{\frac{V}{z}} = \sqrt{\frac{15}{8}}\, y$$

$$\sqrt{\frac{8}{15}}\, x \overset{\times\sqrt{15}}{\underset{\div\sqrt{8}}{}} = \sqrt{\frac{15}{8}}\, y \overset{\times\sqrt{15}}{\underset{\div\sqrt{8}}{}}$$

$$x = \frac{15}{8} y \quad \therefore \quad y = \frac{8}{15} x.$$

We now make z vary.

$$V = xyz$$

$$V = x \times \frac{8}{15} x \times z$$

$$A(x) = (x + z + \tfrac{1}{2} z)(2y + 2z + \tfrac{1}{2} y).$$

$$V = \frac{8}{15} x^2 \times z$$

$$A(x) = \left(x + \tfrac{3}{2} z\right)\left(2z + \tfrac{5}{2} y\right)$$
$$= \left(x + \tfrac{3}{2} z\right)\left(2z + \tfrac{5}{2}\cdot\tfrac{8}{15} x\right)$$
$$= \left(x + \tfrac{3}{2} z\right)\left(2z + \tfrac{4}{3} x\right)$$
$$= 2xz + \tfrac{4}{3} x^2 + 3z^2 + 2xz$$
$$= \tfrac{4}{3} x^2 + 4xz + 3z^2$$
$$= \tfrac{4}{3} x^2 + 4x \frac{V}{\frac{8}{15} x^2} + 3\left(\frac{V}{\frac{8}{15} x^2}\right)^2$$
$$= \tfrac{4}{3} x^2 + \frac{4V}{\frac{8}{15} x} + \frac{3V^2}{\frac{64}{225} x^4}$$
$$= \tfrac{4}{3} x^2 + 4V \tfrac{15}{8} x^{-1} + 3V^2 \tfrac{225}{64} x^{-4}$$
$$A(x) = \tfrac{4}{3} x^2 + \tfrac{15}{2} V x^{-1} + \tfrac{675}{64} V^2 x^{-4}$$
$$A'(x) = \tfrac{8}{3} x - \tfrac{15}{2} V x^{-2} - \tfrac{675}{16} V^2 x^{-5} = 0$$
$$= \tfrac{8}{3} x - \frac{\frac{15}{2} V}{x^2} - \frac{\frac{675 v^2}{16}}{x^5} = 0 \quad (\times \text{ by } x^2)$$
$$\tfrac{8}{3} x^3 - \tfrac{15}{2} V - \frac{\frac{675}{16} v^2}{x^3} = 0 \quad (\times \text{ by } x^3)$$
$$= \tfrac{8}{3} x^6 - \tfrac{15}{2} V x^3 - \tfrac{675}{16} v^2 = 0$$

Let $a = x^3$

$$\tfrac{8}{3} a^2 - \tfrac{15}{2} Va - \tfrac{675}{16} v^2 = 0$$

To solve use quadratic formula to find a.

$a = 8 \quad b = -15 \quad c = -675v^2$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{- \frac{-15}{2} v \pm \sqrt{\left(\frac{-15}{2} v\right)^2 - 4\left(\frac{8}{3}\right)\left(\frac{-675}{16} v^2\right)}}{2 \times 8/3}$$

$$= \frac{\frac{15}{2} v \pm \sqrt{\frac{225}{4} v^2 + 450 v^2}}{16/3}$$

$$= \frac{\frac{15}{2} v \pm \sqrt{\frac{2025}{4} v^2}}{16/3}$$

$$a = \frac{\frac{15}{2} v + \frac{45}{2} V}{16/3}$$

$$= \frac{30V}{16/3}$$

$$a = \frac{45}{8} V$$

$$x^3 = \frac{45}{8} V$$

$$x = \sqrt[3]{45/8 \, V}$$

$$y = \frac{8}{15} x$$

$$= \frac{8}{15} \times \sqrt[3]{\frac{45}{8} V}$$

$$z = \frac{V}{8/15 \, x^2}$$

$$z = \frac{V}{\frac{8}{15} \times \left(\frac{15}{8} V\right)^{2/3}}$$

$$a = \frac{\frac{15}{2} V - \frac{45}{2} V}{16/3}$$

$$= \frac{-15V}{16/3}$$

$$a = \frac{-45}{16} V$$

$$x^3 = \frac{-45}{16} V$$

$$x = \sqrt[3]{\frac{-45}{16} V}$$

NOTE: cannot be worked out as z cannot be a $-$ve value.

$$z = \frac{V}{\frac{8}{15} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times V^{\frac{2}{3}}}$$

$$z = \frac{V^{\frac{1}{3}}}{\frac{8}{15} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \left(\frac{45}{8}\right)^{\frac{1}{3}}}$$

To find z in terms of x:

$$bx = z$$

$$b\sqrt{\frac{45}{8}V} = \frac{V^{\frac{1}{3}}}{\frac{8}{15} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \left(\frac{45}{8}\right)^{\frac{1}{3}}}$$

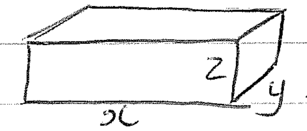
$$b = \frac{\cancel{V^{\frac{1}{3}}}}{\frac{8}{15} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \left(\frac{45}{8}\right)^{\frac{1}{3}} \times \cancel{V^{\frac{1}{3}}}}$$

$$= \frac{1}{\frac{8}{15} \times \frac{45}{8}}$$

$$= \frac{1}{3} \quad \therefore \quad z = \frac{x}{3}$$

$$x : \frac{8}{15}x : \frac{x}{3}$$

Appendix ⑥:



Length of paper $= x + z + 2k_1$

Width $= 2y + 2z + 2k_2$.

$2(y + z + k_2)$

Let V, z be fixed. $\quad V = xyz \quad xy = \frac{V}{z}$

$y = \frac{V}{zx}$.

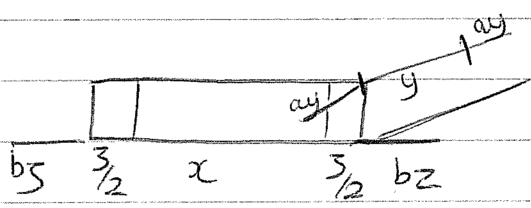
$A = 2(x + z + 2k_1)(y + z + k_2)$

$= 2(xy + xz + xk_2 + zy + z^2 + zk_2 + 2k_1y + 2k_1z + 2k_1k_2)$.

$A = 2xy + 2xz + 2xk_2 + 2zy + 2z^2 + 2zk_2 + 4k_1y + 4k_1z + 4k_1k_2)$

cannot reduce to one variable, ∴ cannot be solved.

Appendix ⑦



Vz is constant

$V = xyz$

$xy = \frac{V}{z}$

$x = \sqrt{\frac{V}{z}} = y$.

Length $= x + z + 2bz$

Width $= 2y + 2z + ay$

Length $= x + z(1 + 2b)$

Width $= 2z + y(2 + a)$

$$A = [x + z(1+2b)][2z + y(2+a)]$$

$$A = \left[\frac{\sqrt{V}}{\sqrt{z}} + z(1+2b)\right]\left[2z + \frac{\sqrt{V}}{\sqrt{z}}(2+a)\right].$$

$$\frac{dA}{dz} = \left[-\tfrac{1}{2}\sqrt{V}(\sqrt{z})^{3/2} + (1+2b)\right]\left[2z + \frac{\sqrt{V}}{\sqrt{z}}(2+a)\right] +$$

$$\left[\frac{\sqrt{V}}{\sqrt{z}} + z(1+2b)\right]\left[2 - \tfrac{1}{2}\sqrt{V}(2+a)z^{-3/2}\right]$$

$= -\sqrt{v}\, z^{-\frac{1}{2}} - \frac{1}{2} v z^{-2}(2+a) + 2z(1+2b) + \frac{\sqrt{v}}{\sqrt{z}}(2+a)(1+2b)$

$+ \frac{2\sqrt{v}}{\sqrt{z}} - \frac{v}{2z^2}(2+a) + 2z(1+2b) - \frac{\sqrt{v}(1+2b)(2+a)}{2z^{\frac{1}{2}}}.$

$= 2z(1+2b) + 2z(1+2b)$

$= -\sqrt{v}\, z^{-\frac{1}{2}} + (2+a)(1+2b)\sqrt{v}\, z^{-\frac{1}{2}} + 2\sqrt{v}\, z^{-\frac{1}{2}} - \frac{1}{2}\sqrt{v}(1+2b)(2+a) z^{-\frac{1}{2}}$

$-\frac{1}{2} v z^{-2}(2+a) - \frac{v}{2} z^{-2}(2+a).$

$= 4z(1+2b) + \sqrt{v}\, z^{-\frac{1}{2}} - \frac{1}{2}\sqrt{v}(1+2b)(2+a) z^{-\frac{1}{2}} + v z^{-2}(2+a) = 0.$

$\Rightarrow 4z^3(1+2b) + \sqrt{v}\, z^{3/2} - \frac{1}{2}\sqrt{v}(1+2b)(2+a) z^{3/2} + v(2+a) = 0.$

$4(1+2b)z^3 + [\sqrt{v} - \frac{1}{2}\sqrt{v}(1+2b)(2+a)]z^{3/2} + v(2+a) = 0$

Let $q = z^{3/2}$

$4(1+2b)q^2 + [\sqrt{v} - \frac{1}{2}\sqrt{v}(1+2b)(2+a)q + v(2+a) = 0$

$q = \frac{-b \pm \sqrt{b^2-4ac}}{2a}.$

$\frac{-(\sqrt{v} - \frac{1}{2}\sqrt{v})(1+2b)(2+a) \pm \sqrt{v - v(1+2b)(2+a) + \frac{v}{4}(1+2b)^2(2+a)^2 - 16(1+2b)(2+a)v}}{8(1+2b)}.$

$\frac{-\sqrt{v} - \frac{1}{2}\sqrt{v}(1+2b)(2+a) \pm \sqrt{4v(4-68(1+b)(2+a) + 1(1+2b)^2(2+a)^2}}{8(1+2b)}$

* could not solve.

$V = n r^2 h$

$h = \frac{V}{\pi r^2}$.

Appendix (8): $A = 2nrh + n\pi r^2$

$n = 1 \qquad n = 2$

$$A = 2\cancel{n}r\frac{V}{\cancel{n}r^2} + n\pi r^2$$

$$A = \frac{2V}{r} + \pi r^2$$

$$A = 2Vr^{-1} + n\pi r^2$$

$$\frac{dA}{dr} = -\frac{2V}{r^2} + 2n\pi r$$

at T.P $\frac{dA}{dr} = 0$

then $2n\pi r = \frac{2V}{r^2}$

$$n\pi r^3 = V$$

$$r^3 = \frac{V}{n\pi} \qquad r = \sqrt[3]{\frac{V}{n\pi}}$$

$$V = n\pi r^3 = nr^2h$$

$$n\pi r = h$$

$n = 2 \qquad\qquad n = 1$

$h = 2r \qquad\qquad h = r$

$r = \frac{h}{2} \qquad\qquad r = h$.

To show area is a minimum,

$\frac{d^2A}{dr^2} = \frac{4V}{r^3} + 2n\pi > 0$ hence minimum area.

Appendix (9): $\dfrac{d^2A}{dr^2} = 8\pi + \dfrac{4V}{r^3} > 0$ hence minimum

Appendix (10): $A = 2\pi r\,(h + 2kr)$

$= 2\pi r\left(\dfrac{V}{\pi r} + 2kr\right)$

$A = \dfrac{2\pi r V}{\pi r^2} + 4\pi k r^2$

$A = 2Vr^{-1} + 4\pi k r^2$

$\dfrac{dA}{dr} = -\dfrac{2V}{r^2} + 8\pi k r$

at T.P $\dfrac{dA}{dr} = 0$

then $-\dfrac{2V}{r^2} + 8\pi k r = 0$

$8\pi k r = \dfrac{2V}{r^2}$

$4\pi k r = \pi h$

$h = 4kr$.

Second Derivative to prove it is a minimum.

$\dfrac{d^2A}{dr^2} = \dfrac{2V}{r^3} + 8\pi k > 0$ hence a minimum.

Appendix (11): $V = \pi r^2 h \Rightarrow h = \dfrac{V}{\pi r^2}$.

$A(r) = (2\pi r + \pi r)(h + 2r + 1)$

$= (2\pi r + \pi r)(h + 3r)$

$= 2\pi r h + 6\pi r^2 + h\pi r + 3\pi r^2$ $\left(\text{Sub } h = \dfrac{V}{\pi r^2}\right)$

$= \dfrac{2\pi r V}{\pi r^2} + 6\pi r^2 + \dfrac{V\cancel{\pi}\cancel{r}}{\cancel{\pi} r^{\cancel{2}}} + 3\pi r^2$

$= \dfrac{2V}{r} + 6\pi r^2 + \dfrac{V}{r} + 3\pi r^2$

$\dfrac{dA}{dr} = 2Vr^{-1} + 6\pi r^2 + Vr^{-1} + 3\pi r^2$

$= -2V \quad + 12\pi r \quad - V \quad + 3\pi r$

$$-2V + 12\pi r^3 - V + 6\pi r^3 = 0$$

$$12\pi r^3 + 6\pi r^3 = 2V + V$$

$$12\pi r^3 + 6\pi r^3 = 3V$$

$$18\pi r^3 = 3V$$

$$18\cancel{\pi} r^3 = 3\cancel{\pi} r^2 h$$

$$18r = 3h$$

$$h = 6r.$$

END OF APPENDICES