Theme: Determining Curves

Your project must be based on, or incorporate, a study in determining curves. Your project should use mathematics appropriate to the focus of Change and Approximation or Extensions (Change and Approximation). You are encouraged to show initiative and be independent in carrying out your project.

General Advice

Often, to solve a mathematical problem, it is necessary to find a particular line or curve which goes through a given set of points. For example, you have probably found the equation of a line which goes through two points. You will remember that only one straight line can be found that goes exactly through two given points. In this case we say that the two points *determine* the line.

One given point and a given slope will also determine a particular line. However one point alone is not sufficient.

Sometimes, but not always, a line can be determined from a point on it and the area that is enclosed between the line and the positive parts of the co-ordinate axes. For example, the line $x+y=2$ (shown in Figure 1) is the only line which passes through (1,1) and encloses an area of two square units with the positive parts of the co-ordinate axes. Therefore these two pieces of information *determine* that line.

However for the point (2,2) and an enclosed area of nine square units, both $x+2y=6$ and $2x+y=6$ can be drawn, so the given information does *not* determine a particular line (see Figure 2).

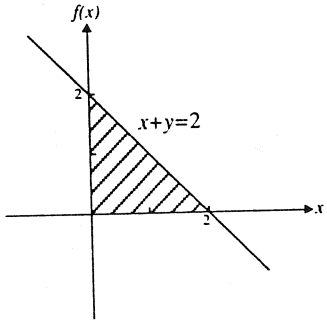

Figure 1. $x+y=2$ is the only line that encloses two square units in the first quadrant.

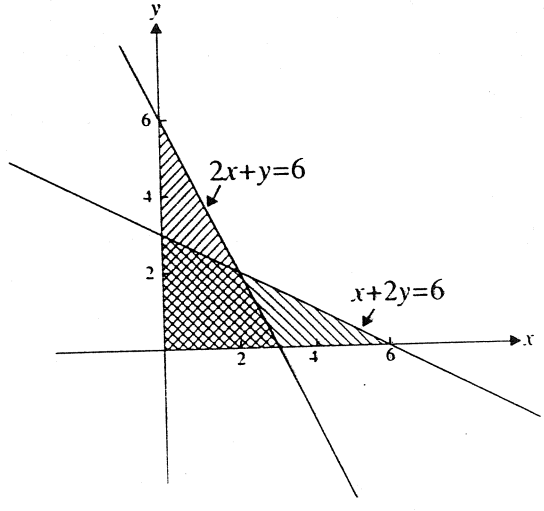

Figure 2. Both $x+2y=6$ and $2x+y=6$ enclose an area of nine square units in the first quadrant

In this project you will explore what *information* can be used to *determine curves* and what *curves* can be found that fit particular sets of information. For example, you may investigate graphs like those shown in Figure 3, all of which pass through the points (-3,0), (0,0) and (5,0)

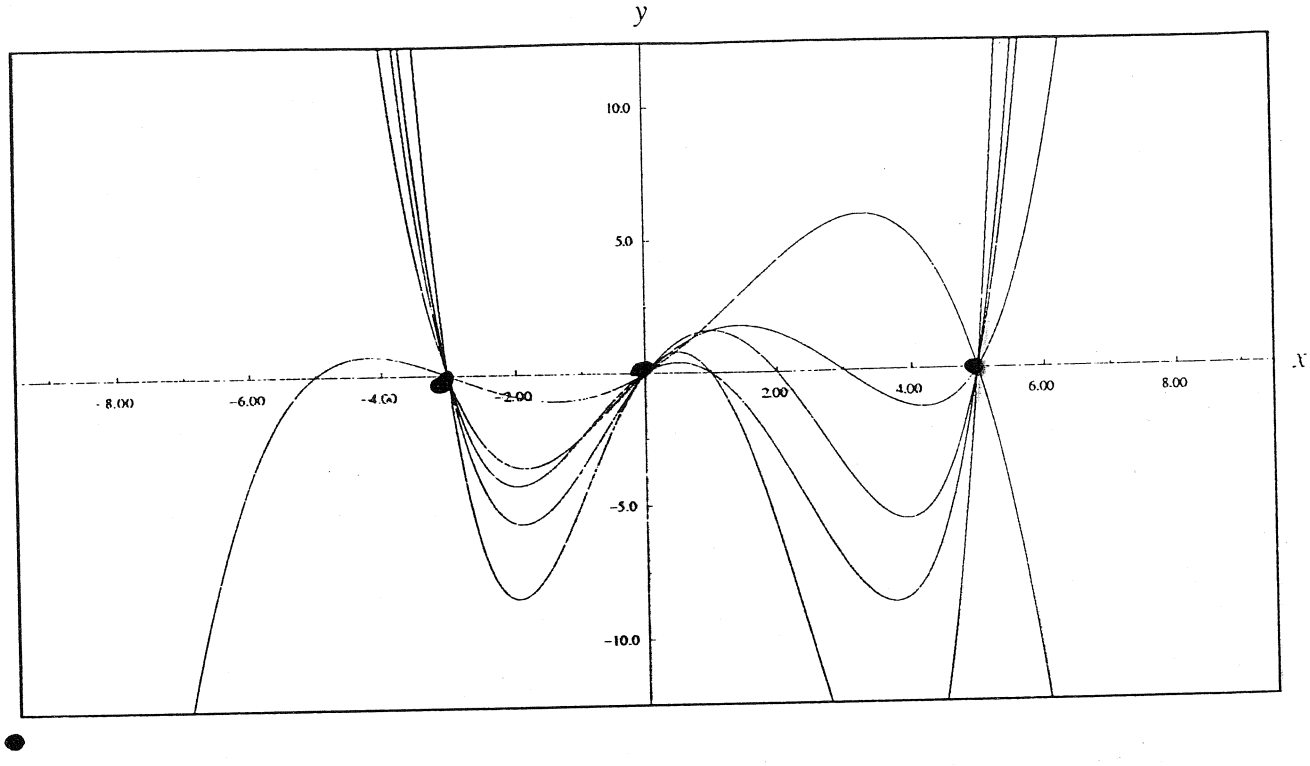

Figure 3. Polynomials of degree 4 pass through the points (-3,0), (0,0) and (5,0)

Types of information you may use to determine curves include: points, gradients, area beneath the curve, symmetry, asymptotes, maxima or turning points. However, you are advised to start by restricting your investigation to simple types of curves (say straight lines or parabolas) and to choose sets of points and other information that make your work easy. You can generalise your work after studying specific cases.

In this advice the word *curve* is used to include straight lines.

Examples of *types of curves* are straight lines, circles, graphs of quadratic equations of the form $y=ax^2+bx+c$, cubic equations, sine functions, and exponential functions.

Remember, to test whether a given set of information *determines a curve* of a given type, you must show two things:

i that there is a curve of that type which fits the information (this usually means finding its equation)

and

ii that no other curve of that type fits the information.

Acknowledge which area of mathematics you use: co-ordinate geometry, calculus or algebra. You may choose to develop a computer program to assist you, or you may use a recognised computer package but remember to include your own analysis of the problem.

Starting Points

You may investigate any topic related to the theme: Determining Curves.

You **must** discuss your choice of topic and how it relates to the theme with your teacher.

Listed below are five possible starting points, but it is not compulsory to use them. However, if you do not use a starting point listed below then you must obtain 'Topic Approval' from your teacher.

Each starting point gives some questions to guide your investigation and indicates directions for extension.

1. Curves through sets of points.

i Consider the three points (1,0), (3,1) and (5,6). Find some curves of different types that go through these three points. Which of these curves are *determined* by the three points?

ii Identify some types of curves which cannot go through all of the above three points.

iii Investigate the problems in *i* and *ii* above for other sets of three points and for sets with more than three points.

iv Now select a type of curve. Identify the properties your sets of points must have so that curves of your selected type can be found to go through them. (For example, you will be able to find a circle through any three points that are not collinear.)

v Investigate *iv* above for other types of curves.

2. How much information?

i Imagine I am thinking of a curve and know its equation. You know that it is the graph of a quadratic equation of the form $y=ax^2+bx+c$, but you do not know the values of a, b and c. What information would enable you to discover exactly which equation I am thinking of? Give a range of examples of sets of information that would identify the equation.

ii How many pieces of information are required? You will have to make your own decision about what one piece of information is. Explain why you have chosen to count information in that way. (For example, if I told you the curve had a minimum value at (3,3) is this one, two or more pieces of information? Why?)

iii Extend this investigation to other types of curves.

3. Motion in a straight line.

If you are given the displacement-time function for a particle moving in a straight line you can calculate displacement, velocity and acceleration at an instant of time. Conversely, information about displacement, velocity and acceleration at a particular time may be used to determine the displacement-time curve for a particle if the type of curve is known.

i Suppose you are given displacement, s; velocity, v; and acceleration, a; at an instant of time, t; say $s=6$, $v=2$, $a=3$ when $t=0$. Choose a type of curve for the displacement-time graph. (eg sine, parabola, exponential, part of a circle). Find all the curves of this type which satisfy the data. Does this information determine the curve? If it is not possible to find a curve which satisfies the information, explain why.

ii Repeat the procedure in *step i* for other types of curves. Compare the motions predicted in each case by drawing the displacement-time graphs on the same axes. In each case describe what a person watching the particle would see, ie the way the particle moves in the line.

iii Repeat *steps i* and *ii* above using different sets of given information. (For example acceleration at two different times and position at another time.)

iv For each set of information, describe the common features of the graphs you have drawn and explain why they occur. (For example, all the curves that fitted the data in the first part went through a common point. Why?)

v Possible extension. If a particle is moving in a plane and not just in a straight line, the motion can be described using a displacement-time graph in each dimension (e.g. horizontal and vertical). Choose sets of data as in *part i* and determine the displacement-time graph for each dimension. Describe what a

4. Families of curves.

Choose a curve of a particular type (say a parabola of the form $y=ax^2+bx+c$ or a circle). Imagine you had been given enough information to determine such a curve but then you lost one piece of information, say one of the points on the curve.

i Draw graphs (on one set of axes) that show the curves that fit the information that is left. These curves belong to a "family of curves". Figure 4, for example, illustrates some of the members of the family of parabolas which pass through (0,6) and have gradient 2 at that point.

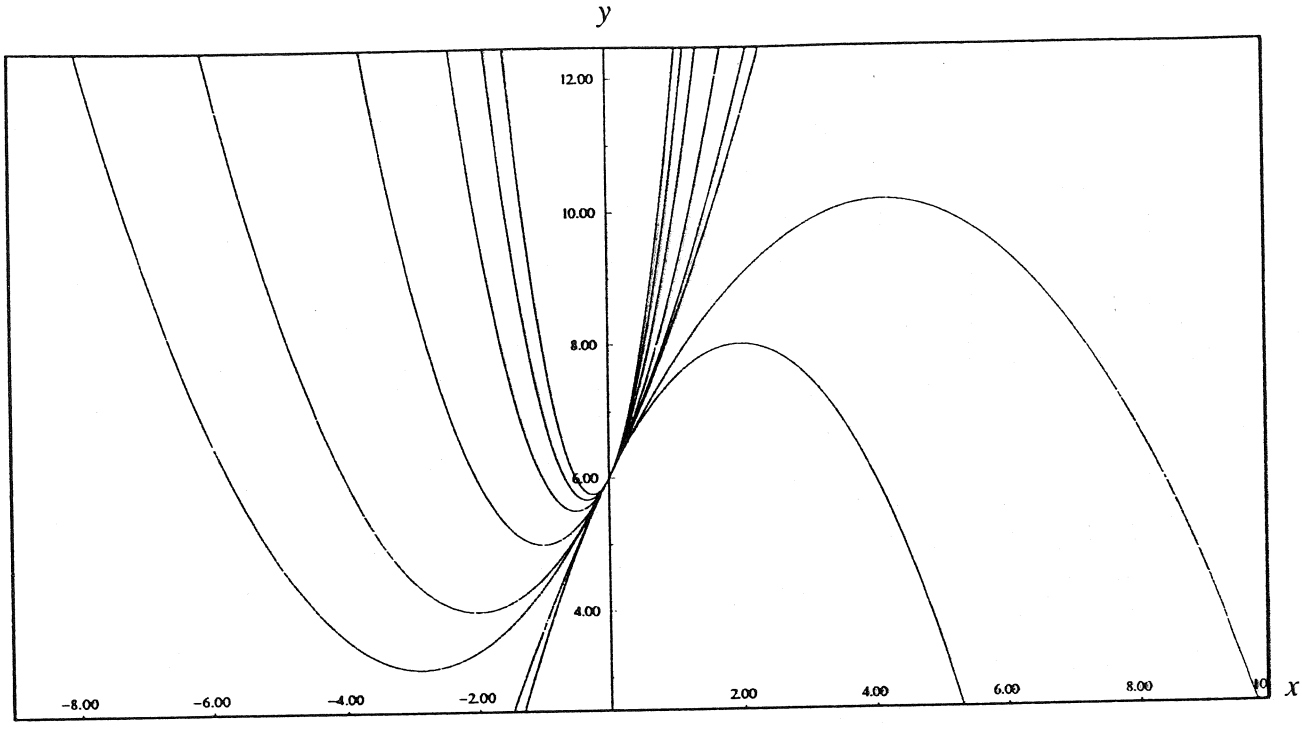

Figure 4.

ii Give some properties of the family of curves that you have sketched.

iii What do the families look like if you lose information of different types, eg information about a point, a gradient, or an asymptote? Illustrate your answers by drawing some families of curves that would result if you lost these pieces of information.

iv Repeat *steps i, ii,* and *iii* looking at other types of curves.

v As an extension, you may like to draw families of curves which are solutions to differential equations and describe their properties. Alternatively you may investigate the questions in *i* to *iv* above for curves given in polar co-ordinates.