INTRODUCTION TO CALCULUS

Calculus can be divided into two key areas:

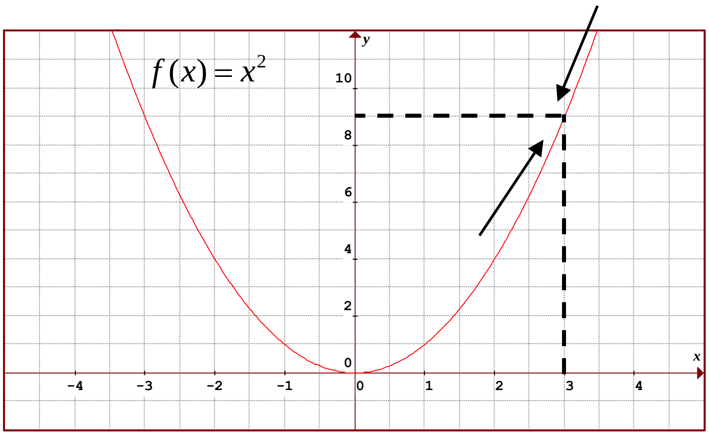
- Differential Calculus – dealing with limits, rates of change, tangents and normals to curves, curve sketching, and applications to maxima and minima problems.

- Integral Calculus, dealing with areas and volumes, and approximate areas under and between curves.

LIMITS

Consider the function $y = f(x)$. The limit of the function is defined as the value of y (ordinate) that is approached by the curve, as the value of x (or abscissa) approaches a particular number, say of value 'a'.

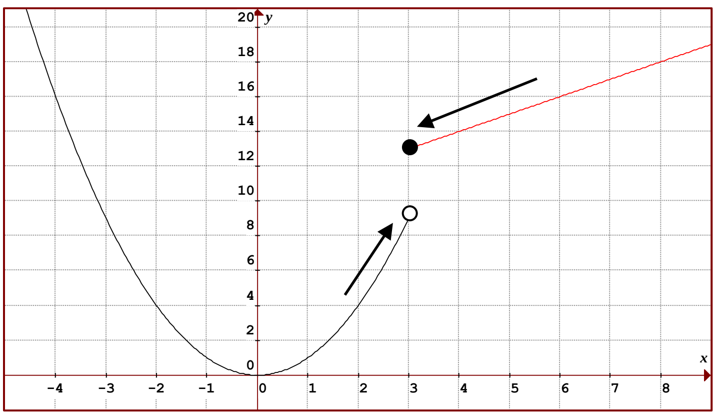
For example, if we approach an x value of 3 from the left or the right, on the curve $f(x) = x^2$ we see that in both cases, the limiting value is 9. i.e. $\lim_{x \to 3} x^2 = 9$.



$f(3) = (3)^2 = 9$

As a further example, the diagram below is a graph of a piecemeal function. Note that one end point is open, and the other closed, above $x = 3$.

$f(x) = x^2$



$f(x) = x + 10$

Regardless, as we approach $x = 3$ from the left, the limiting value is 9, but as we approach $x = 3$ from the right, the limiting value is 13. Hence we say that the limit here does not exist. i.e. $\lim_{x \to 3} f(x)$ does not exist. Limits (and hence tangents) can only be found on curves that are continuous at the point in question.

Note: Do not confuse this with the functional value $f(3)$, which in this case is clearly defined as 13.

EVALUATING LIMITS

In each of the cases below, you should initially make the substitution of the given x value, in an attempt to evaluate the limit.

Depending on what results from this then determines the necessary strategy. Note that in all cases, the use of a calculator to substitute a number very close to the given x value will give you a very good idea of what to expect as a final answer – very handy in an exam if you forget the formal method!

CASE I: You substituted into the expression, and the result was well defined. In other words, the resulting expression could be evaluated – do so!

QUESTION 18

Find $\lim_{x \to 2}(4x-1)$.

Solution

$$\begin{aligned}\lim_{x \to 2}(4x-1) &= 4(2)-1\\ &= 7\end{aligned}$$

QUESTION 19

Find $\lim_{x \to -1}\left(\frac{x^2}{5-x}\right)$.

Solution

$$\begin{aligned}\lim_{x \to -1}\left(\frac{x^2}{5-x}\right) &= \frac{(-1)^2}{5-(-1)}\\ &= \frac{1}{6}\end{aligned}$$

The last expression was a "Quotient". Quotients of expressions often create difficulty. Imagine you substituted into such an expression, and the result was one of the following...

CASE II:

$$\frac{Zero}{Constant} \left(i.e. \frac{0}{c} \right)$$

The limit is 0.

QUESTION 20

Find $\lim\limits_{x \to 5} \left(\frac{x-5}{x+4} \right)$.

Solution

$$\lim_{x \to 5} \left(\frac{x-5}{x+4} \right) = \frac{(5)-5}{(5)+4}$$

$$= \frac{0}{9}$$

$$= 0$$

CASE III:

$$\frac{Constant}{Zero} \left(i.e. \frac{c}{0} \right)$$

The limit is undefined since it approaches $\pm \infty$.

QUESTION 21

Find $\lim\limits_{x \to -2} \left(\frac{x^2}{x+2} \right)$.

Solution

$$\lim_{x \to -2} \left(\frac{x^2}{x+2} \right) = \frac{(-2)^2}{(-2)+2}$$

$$= \frac{4}{0}$$

$$\to \pm\infty$$

CASE IV:

$$\frac{Infinity}{Constant}\ \left(i.e.\ \frac{\infty}{c}\right)$$

The limit is undefined since it approaches $\pm\infty$.

QUESTION 22

Find $\lim\limits_{x\to\infty}\left(\dfrac{9+x}{7}\right)$.

Solution

$$\lim_{x\to\infty}\left(\frac{9+x}{7}\right)=\frac{9+(\infty)}{7}$$

$$=\frac{\infty}{7}$$

$$\to\infty$$

CASE V:

$$\frac{Constant}{Infinity}\ \left(i.e.\ \frac{c}{\infty}\right)$$

The limit is 0.

QUESTION 23

Find $\lim\limits_{x\to\infty}\left(\dfrac{8}{x}\right)$.

Solution

$$\lim_{x\to\infty}\left(\frac{8}{x}\right)=\frac{8}{(\infty)}$$

$$=0$$

CASE VI:

$$\frac{Zero}{Zero}\ \left(i.e. \frac{0}{0}\right)$$

You should simplify the algebraic fraction by first factoring, and then cancelling, before re-substituting the given x value.

QUESTION 24

Find $\lim\limits_{x \to -1} \left(\dfrac{x^2 - 3x - 4}{10x + 10}\right)$.

Solution

$$\lim_{x \to -1} \left(\frac{x^2 - 3x - 4}{10x + 10}\right)$$

CASE VII:

$$\frac{Infinity}{Infinity}\ \left(i.e. \frac{\infty}{\infty}\right)$$

Divide every term in the expression by the highest power of x present in the denominator, simplify each term, then use the rule for Case V.

QUESTION 25

Find $\lim\limits_{x \to \infty} \left(\dfrac{x^2 - 5}{8 + 2x + 3x^2}\right)$. ($x^2$ is the highest power of x in the denominator).

Solution

$$\lim_{x \to \infty} \left(\frac{x^2 - 5}{8 + 2x + 3x^2}\right)$$

WATCH OUT!

If ever in doubt about the correct technique, or in doubt of your answer, simply substitute a value for x that is close to the given value, into the given formula, on your calculator. For example, for $\lim_{x \to 3}$ say, use $x = 2.99$ or 3.01, and for $\lim_{x \to \infty}$, use some large value like $x = 1{,}000{,}000$.

THE DERIVATIVE OF A FUNCTION

In graphing, you have used tables of values, to plot points (along with other algebraic tools such as finding any axes intercepts etc) and to draw sketches of functions. The original function $y = f(x)$ is used to find the y values or ordinates of points, on the graph, so that we can see its shape and position.

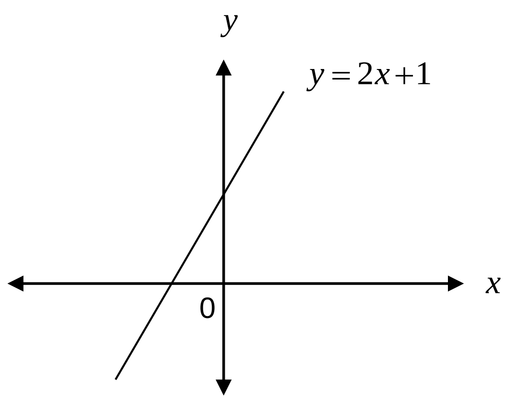
The "derivative", or derived function is written as $\frac{dy}{dx}$ or $f'(x)$. As we are assuming here that the independent variable in the function is x, then the derivative is found "with respect to x".

The derivative is related to, and found from, the original function $y = f(x)$. The derivative is used to find a different feature of the graph – it is used to find the gradient.

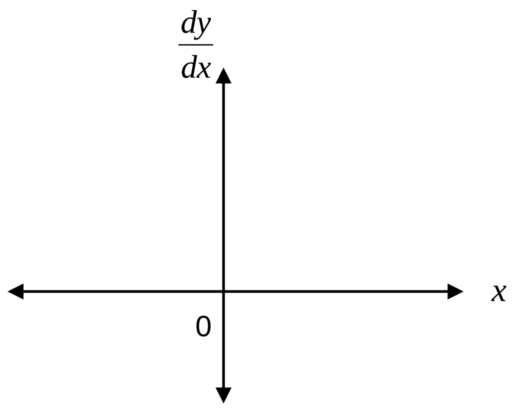
The process of finding the derivative is called "differentiation".

QUESTION 26

Sketch the derived function $\frac{dy}{dx}$ for the given function, y.

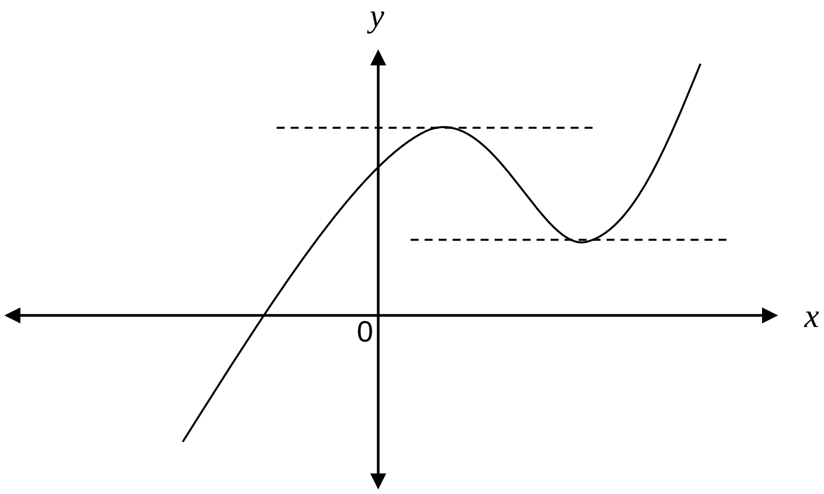


Solution

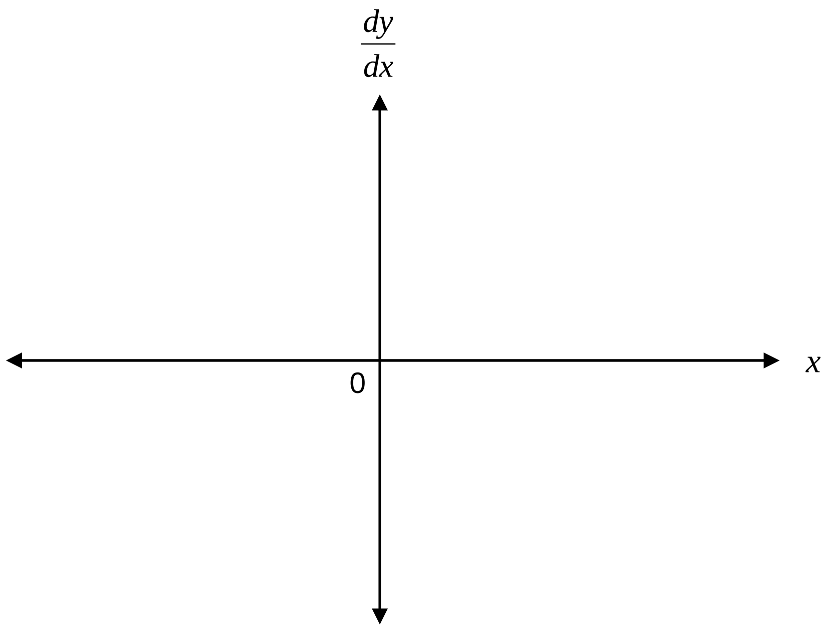


QUESTION 27

Sketch the derived function $\frac{dy}{dx}$ for the given function, y.

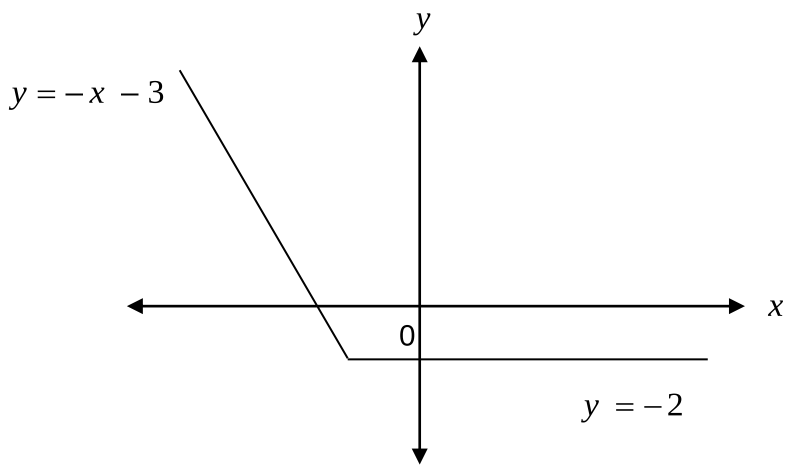


Solution

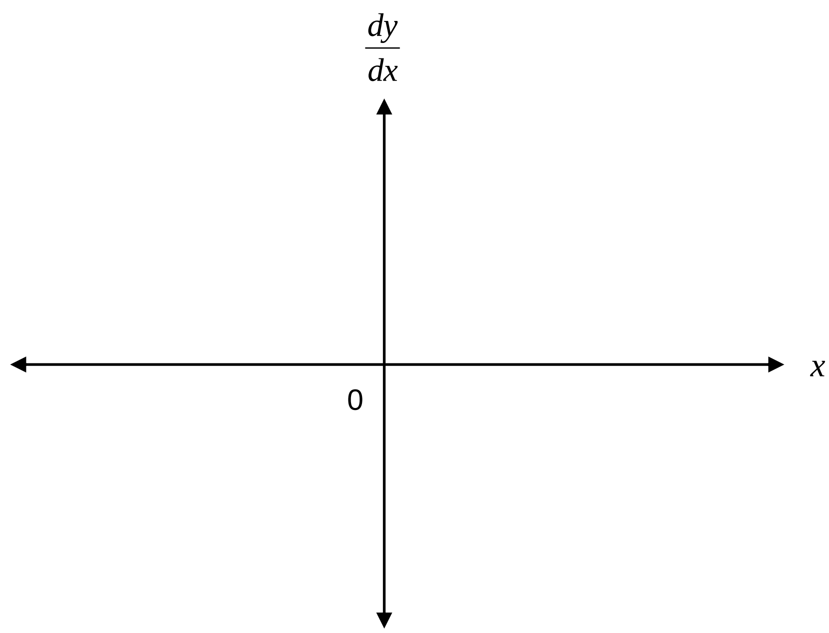


QUESTION 28

Sketch the derived function $\frac{dy}{dx}$ for the given function, y.



Solution



DIFFERENTIABILITY

A function is said to be "differentiable" at a particular point if it satisfies two conditions – the graph of the function at the point in question must be continuous (the curve is not broken at the point), and the graph at the point must also be smooth (the curve must have the same slope on the left hand approach to the point as it does on the right hand approach).

ESTIMATING THE GRADIENT OF THE TANGENT

If we had an accurate graph of a function, we could draw a tangent at the point in question, judging this purely by eye.

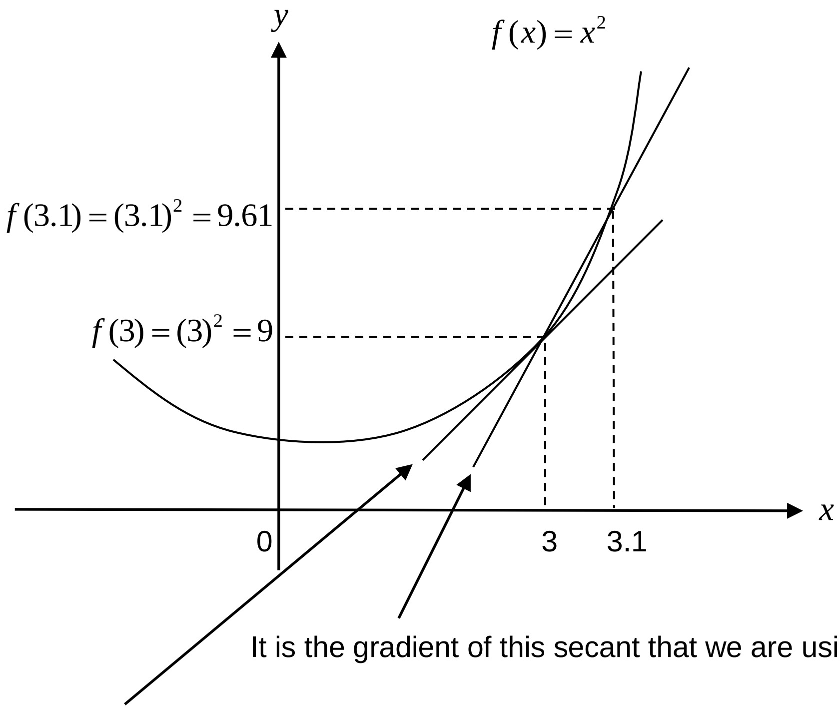
By choosing two points on the tangent and reading off the coordinates of the two points, we would use the gradient formula $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ to calculate the gradient of the tangent at that particular point.

If we had a rule for the function, without an accurate graph, we could use the following method.

Say we wanted the gradient of $f(x) = x^2$ at the point where $x = 3$.

We could use a second point on the curve, very close to the first and then apply the gradient formula.

Let's use the two points at $x = 3$ and $x = 3.1$. Note that this cannot give the exact gradient at $x = 3$ - just something close.



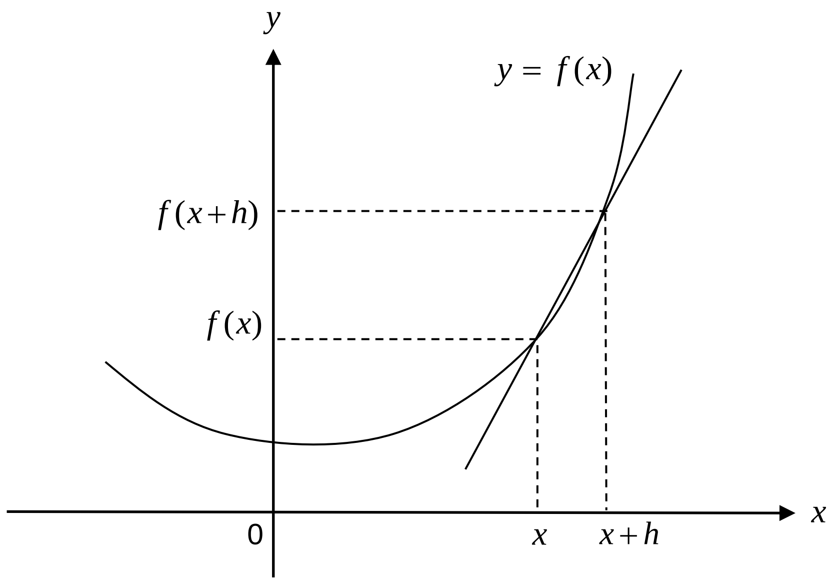
It is the gradient of this secant that we are using.

This is what we really want – the gradient of the tangent. This is because the slope of the tangent and the slope of the curve are the same.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9.61 - 9}{3.1 - 3} = \frac{0.61}{0.1} = 6.1$$

This value represents the gradient of the secant joining the two points, and not of the curve or tangent. But we might guess that the gradient is around 6, and in fact we will shortly find out it is exactly 6.

DIFFERENTIATION USING FIRST PRINCIPLES



The gradient of the secant is given by:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{(x+h) - x}$$

Hence the gradient of the tangent (and hence the curve) is:

$$\boxed{\frac{dy}{dx} \;=\; f'(x) \;=\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

METHOD:

Step 1: Write the expressions for $f(x)$ and $f(x+h)$.

Step 2: Substitute the expressions for $f(x)$ and $f(x+h)$ into the limit theorem.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

Step 3: Expand and collect like terms.

Step 4: Remove h as a common factor and simplify.

Step 5: Substitute $h = 0$.

Note: To find the gradient at a specific point, we substitute the value of x into the derivative.

Note: The gradient at $x = 2$ is denoted as $f'(2)$.

QUESTION 29
Find the derivative, using first principles, of $f(x)=x^2$.

Solution

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-x^2}{h}$$

$$=\lim_{h\to 0}\frac{2xh+h^2}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$=2x+0$$

$$=2x$$

This answer is called the derivative or derived function or gradient function of $f(x)$.

Note that at $x=3$, $f'(3)=2(3)=6$. This is the same answer as that guessed in the earlier example.

QUESTION 30
Find, using first principles, the gradient function of $f(x)=x^2-5x+2$.

Solution

<u>WATCH OUT!</u>
When the function has two terms or more, make sure that the function is written inside of brackets behind the minus sign.

$$f(x)=x^2-5x+2$$

$$f(x+h)=(x+h)^2-5(x+h)+2$$

$$=x^2+2xh+h^2-5x-5h+2$$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad =\lim_{h\to 0}\frac{(x^2+2xh+h^2-5x-5h+2)-(x^2-5x+2)}{h}$$

$$=\lim_{h\to 0}\frac{x^2+2xh+h^2-5x-5h+2-x^2+5x-2}{h} \qquad =\lim_{h\to 0}\frac{2xh+h^2-5h}{h}$$

$$=\lim_{h\to 0}\frac{h(2x+h-5)}{h} \qquad =\lim_{h\to 0}(2x+h-5) \qquad =2x+(0)-5 \qquad =2x-5$$

DIFFERENTIATION RULES

The derivative of an algebraic term is obtained by multiplying the term by the power and then lowering the power on that term by one. This rule applies for all algebraic expressions, including rational functions, providing that $n \neq 0$.

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

For example: If $y = 2x^6$ then $\frac{dy}{dx} = 2 \times 6 \times x^{6-1} = 12x^5$

Note also that if $y = ax$ then $\frac{dy}{dx} = a$.

The derivative of a constant (a term that does not contain any variables) is equal to zero.

If $y = c$ (where c is a constant) then $\frac{dy}{dx} = 0$

The derivative of a sum (or of a difference) is the sum (or differences) of the individual derivatives.

If $y = u(x) \pm v(x)$ then $\frac{dy}{dx} = u'(x) \pm v'(x)$

QUESTION 31

Find the derivative of $y = x^2 - 5x + 2$.

Solution

WATCH OUT!

"d" for differentiate, "d" for decrease the power.
(Please note that this is only true for a power which is a constant only)

$$\frac{dy}{dx} = \frac{d(x^2)}{dx} - \frac{d(5x)}{dx} + \frac{d(2)}{dx}$$

$$= 2x - 5 + 0$$

$$= 2x - 5$$

FINDING DERIVATIVES

Step 1: Rewrite all terms as powers of x.

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \sqrt[3]{x} = x^{\frac{1}{3}} \qquad (\sqrt[p]{x})^q = (x^{\frac{1}{p}})^q = x^{\frac{q}{p}}$$

Step 2: Bring terms involving x in the denominator (bottom of a fraction) to the top, by changing the sign on the power.

For example: $\dfrac{1}{x^2} = x^{-2}$

Note: $\dfrac{1}{6x^2} = \dfrac{x^{-2}}{6}$

Step 3: Simplify expressions so that terms are separated by addition and subtraction and then differentiate each term individually. Alternatively, reduce expressions down to 1 term only (use log and index laws).

Products:

Expand simple products rather than applying the Product Rule.

Quotients:

- Remove common factor(s) and simplify.
- Factorise and eliminate terms by cancellation.
- If there is only one term in the denominator, write each term in the numerator over the denominator so that individual fractions are formed. Then simplify each term by cancellation.

For example: $\dfrac{x^2+1}{x^2} = \dfrac{x^2}{x^2} + \dfrac{1}{x^2} = 1 + x^{-2}$ (Don't use the Quotient Rule).

Step 4: Differentiate.

Step 5: Re-write the answer using positive powers. Bring terms with negative powers in the numerator (top of a fraction) to the bottom, by changing the sign on each power.

Step 6: State restrictions on the values of x.

QUESTION 32

Differentiate $y = 16.3x^{4.21} + 2.5x^{-0.8}$.

Solution

$$\frac{dy}{dx} = 16.3 \times 4.21x^{3.21} + 2.5 \times -0.8x^{-1.8}$$
$$= 68.623x^{3.21} - 2x^{-1.8}$$

QUESTION 33

Find the derivative of $y = ax^2 + bx + c$.

Solution

$$\frac{dy}{dx} = a \times 2x^1 + b \times 1x^0 + 0$$
$$= 2ax + b$$

QUESTION 34

Differentiate $y = 4x + 3 - \frac{5}{x} + \frac{2}{x^3} \qquad x \neq 0$.

Solution

Bring terms involving x in the denominator (bottom of a fraction) to the top by changing the sign on the power:

$$y = 4x + 3 - 5x^{-1} + 2x^{-3}$$

Differentiate each term individually and add/subtract the results:

$$\frac{dy}{dx} = 4 + 5x^{-1-1} + {}^{-}\,6x^{-3-1} = 4 + 5x^{-2} - 6x^{-4}$$
$$= 4 + \frac{5}{x^2} - \frac{6}{x^4}$$

QUESTION 35

Differentiate $y = 2x + \frac{3}{\sqrt{x}} - 3\sqrt{x}$.

Solution

Rewrite all terms as powers on x:

$$y = 2x + \frac{3}{x^{\frac{1}{2}}} - 3x^{\frac{1}{2}}$$

Bring terms involving x in the denominator (bottom of a fraction) to the top by changing the sign on the power:

$$y = 2x + 3x^{\frac{-1}{2}} - 3x^{\frac{1}{2}}$$

Differentiate each term individually and add/subtract the results:

$$\frac{dy}{dx} = 2 - \frac{3}{2}x^{\frac{-3}{2}} - \frac{3}{2}x^{\frac{-1}{2}}$$

$$\frac{dy}{dx} = 2 - \frac{3}{2x^{\frac{3}{2}}} - \frac{3}{2x^{\frac{1}{2}}}$$

QUESTION 36

Differentiate $y = \frac{x^2 - 5x + 6}{x - 3}$ with respect to x.

Solution

Factorise the numerator and eliminate terms by cancellation:

$$y = \frac{(x-3)(x-2)}{(x-3)} = x - 2$$

Differentiate each term individually and add/subtract the results:

$$\frac{dy}{dx} = 1$$

QUESTION 37

Differentiate $\dfrac{x^3 - x^2}{3x}$ with respect to x.

Solution

Write each term in the numerator over the denominator so that individual fractions are formed. Then simplify each term by cancellation:

$$y = \frac{x^3 - x^2}{3x} = \frac{x^3}{3x} - \frac{x^2}{3x} = \frac{x^2}{3} - \frac{x}{3}$$

Differentiate each term individually and add/subtract the results:

$$\frac{dy}{dx} = \frac{2x}{3} - \frac{1}{3}$$

QUESTION 38

Differentiate $y = (8x+1)(5 - x^2)$.

Solution

We cannot differentiate a product, if left as a product, using the rules we have seen so far (or a quotient either).

Therefore, we expand simple products before differentiating. (If it is impractical to expand a product, we can use a special "Product Rule".)

$$\begin{aligned} y &= (8x+1)(5-x^2) \\ &= 40x - 8x^3 + 5 - x^2 \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= 40 \times 1x^0 - 8 \times 3x^2 + 0 - 2 \times x^1 \\ &= 40 - 24x^2 - 2x \end{aligned}$$

THE PRODUCT RULE

The product rule is used to find the derivative of the product of two functions that cannot be simplified by standard algebraic techniques.

The Product Rule states if $y = u(x).v(x)$ then the derivative is equal to:

$$\frac{dy}{dx} = v.\frac{du}{dx} + u.\frac{dv}{dx}$$

An alternative notation is: $y' = v.u' + u.v'$

where $v' = \frac{dv}{dx}$ **and** $u' = \frac{du}{dx}$

METHOD:

Step 1: Let the second function equal v. Find the derivative $\frac{dv}{dx}$.

Step 2: Let the first function equal u. Find the derivative $\frac{du}{dx}$.

Step 3: Substitute the relevant derivatives and equations into the product rule.

WATCH OUT!

Write down your u and $\frac{du}{dx}$, your v and $\frac{dv}{dx}$, first, before substituting into your "Product Rule" (ditto for the upcoming "Quotient Rule").

QUESTION 39

Differentiate $y = \left(x^3 - 6\right)\left(5x^2 + 3x - 1\right)$.

Solution

$u = x^3 - 6$ $\qquad$ $v = 5x^2 + 3x - 1$

$\frac{du}{dx} = 3x^2$ $\qquad$ $\frac{dv}{dx} = 10x + 1$

$$\frac{dy}{dx} = (5x^2 + 3x - 1).3x^2 + (x^3 - 6).(10x + 3)$$

$$= 15x^4 + 9x^3 - 3x^2 + 10x^4 + 3x^3 - 60x - 18$$

$$= 25x^4 + 12x^3 - 3x^2 - 60x - 18$$

(With examples like the above, it is sometimes better to expand the original function first, then differentiate, without using the product rule. However, future examples will see the benefit of using the product rule).

THE QUOTIENT RULE

The Quotient Rule is used to differentiate the quotient of two functions that cannot be simplified by standard algebraic techniques.

The Quotient Rule states that if $y = \frac{u(x)}{v(x)}$ then $\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$.

An alternative notation is: $y' = \frac{v.u' - u.v'}{v^2}$

where $v' = \frac{dv}{dx}$ **and** $u' = \frac{du}{dx}$

METHOD:

Step 1: Let the equation in the numerator equal u. Find the derivative $\frac{du}{dx}$.

Step 2: Let the equation in the denominator equal v. Find the derivative $\frac{dv}{dx}$.

Step 3: Substitute the derivatives and equations into the Quotient Rule.

Step 4: Simplify.

QUESTION 40

Find the derivative of $y = \frac{x}{x^2+1}$.

Solution

Let the function in the numerator equal u: Let $u = x$

Find the derivative $\frac{du}{dx}$**:** $\frac{du}{dx} = 1$

Let the function in the denominator equal v: Let $v = x^2 + 1$

Find the derivative $\frac{dv}{dx}$**:** $\frac{dv}{dx} = 2x$

Substitute the derivatives and relevant equations into the quotient rule and simplify:

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

$$\therefore \frac{dy}{dx} = \frac{(x^2+1)1 - x.2x}{(x^2+1)^2} = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

THE CHAIN RULE

The Chain Rule is used to differentiate complex expressions involving one equation in terms of x. The Chain Rule is also known as the "function-of-a-function rule", or the "composite function rule".

The Chain Rule states that if $y = u[v(x)]$ then $\frac{dy}{dx} = \frac{du}{dv} \times \frac{dv}{dx}$

This is more simply stated as "...the derivative about the outside of the bracket multiplied by the derivative of the expression inside of the bracket ..."

QUESTION 41

Differentiate $y = (1-2x)^9$ with respect to x.

Solution

$$\frac{dy}{dx} = 9(1-2x)^{9-1}.-2$$
$$= -18(1-2x)^8$$

QUESTION 42

Differentiate $y = \sqrt[3]{6x^4 + 1}$ with respect to x.

Solution

Rewrite this in index form first:

$$y = (6x^4 + 1)^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \tfrac{1}{3}(6x^4 + 1)^{\frac{1}{3}-1}.24x^3$$

$$= 8x^3(6x^4 + 1)^{-\frac{2}{3}}$$

$$= \frac{8x^3}{(6x^4 + 1)^{\frac{2}{3}}}$$

$$= \frac{8x^3}{\sqrt[3]{(6x^4 + 1)^2}}$$

QUESTION 43

If $f(x) = (2x+5)(9-x)^4$ find x such that $f'(x) = 0$.

Solution