



UNIT 3 FURTHER MATHEMATICS SUMMARY NOTES FOR THE VCAA EXAMS

**WRITTEN BY A 99.95 STUDENT WHO
OBTAINED A 47 STUDY SCORE**

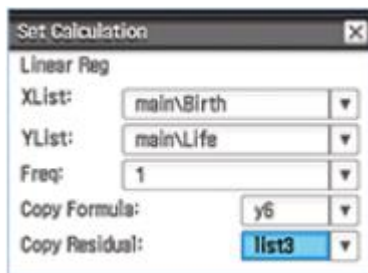
Data Analysis

Regression Analysis

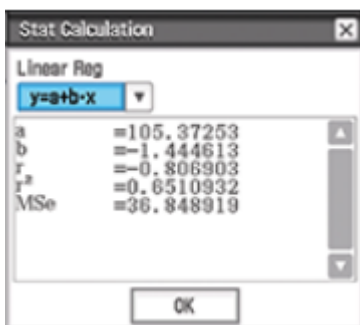
1. Write down the EV and RV names as the list names
2. Enter data into lists
3. Construct a scatterplot (Set Graph)



4. Find the Least squares regression line (Calc > Regression > Linear Reg)



5. Write down the key results and graph residuals against EV to test linearity assumption



Univariate Data

Categorical Variables: represents characteristics or qualities of people or things

- **Nominal:** data values that can be used to group individuals according to a characteristic
Example. Eye Colour, Gender, Postal Code
- **Ordinal:** data values that can be used to both **group** and **order** individuals according to a characteristic
Example. Fitness Level, Economic Status, Education Level

Numerical Variables: represents quantities and things that can be counted or measured

- **Discrete:** represents quantities that are counted in exact values
Example. Number of People, Pages in a book, Goals scored
- **Continuous:** represents quantities that are measured on a decimal scale
Example. Weight, Temperature, Costs to fill a tank with petrol

Frequency Table

- A listing of values a variable takes in a dataset, along with how frequently each value occurs

Example:

The sex of 11 preschool children is as shown (F = Female, M = Male):

F M M F F M F F F M M

Sex	Frequency	
	Number	Percentage
Female	6	54.5
Male	5	45.5
Total	11	100.0

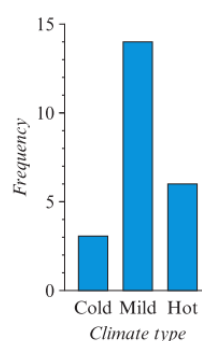
Bar Chart

- Represents key information in a frequency tables as picture, which has bars of equal width and spacing to represent each category
- Note:** may be frequency or percentage frequency

Example:

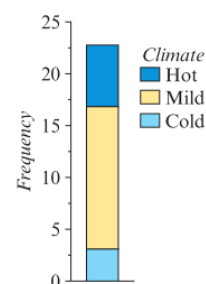
The climate type of 23 countries is classified as 'cold', 'mild', or 'hot'. Construct a frequency bar chart to display this information using the data summarised in the table.

Climate type	Frequency	
	Number	Percentage
Cold	3	13.0
Mild	14	60.9
Hot	6	26.1
Total	23	100.0



Segmented Bar Chart

- Bars that are stacked on top of one another to give a single bar with several segments, with the length of each being the frequency
- Legend is required to identify categories



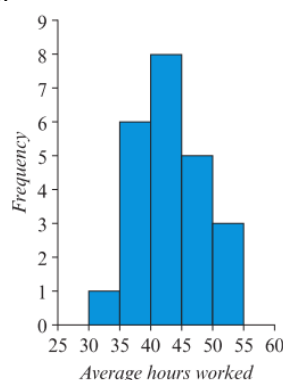
Histograms

- A graphical display of information in a grouped frequency table with bars of equal width and no spacing

Example:

Construct a histogram for the frequency table.

Average hours worked	Frequency
30.0–34.9	1
35.0–39.9	6
40.0–44.9	8
45.0–49.9	5
50.0–54.9	3
Total	23



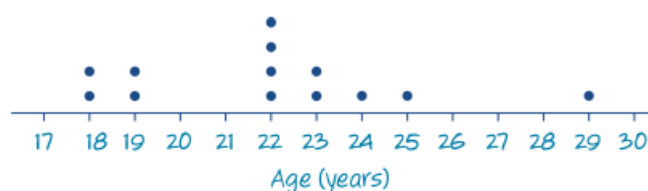
Dot Plots

- Displays discrete numerical data for small data sets

Example:

The ages (in years) of the 13 members of a cricket team are:

22 19 18 19 23 25 22 29 18 22 23 24 22



Stem Plots

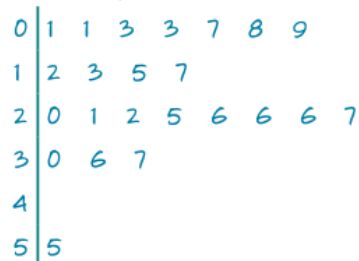
- Displays discrete and continuous data for small to medium sized data sets

Example:

University participation rates (%) in 23 countries are given below.

26 3 12 20 36 1 25 26 13 9 26 27 15 21 7 8 22 3 37 17 55 30 1

Rate (%) Key: 1|2 = 12



Which graph?

Categorical	Bar chart	5-10 categories
	Segmented bar chart	Not too many categories (maximum 4-5)
Numerical	Histogram	Medium – large data sets ($n \geq 40$)
	Stem plot	Small – medium data sets ($n \leq 50$)
	Dot plot	Small data sets ($n \leq 20$)

Median, Range and Interquartile Range

Median: middle value of the ordered data set

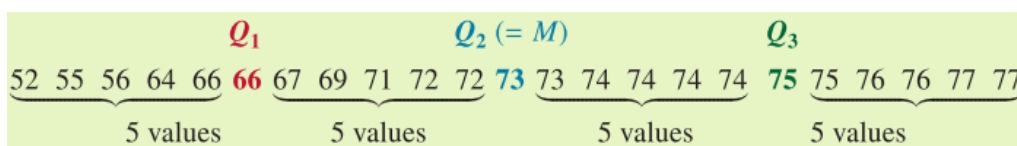
- Located at the $\left(\frac{n+1}{2}\right)th$ position, where n = number of data values
- Measure of centre of a distribution

Range: difference between the largest and smallest value in the data set

- R = largest data value – smallest data value
- Measure of spread of a distribution, the maximum spread of the data values

Interquartile Range: the spread of the middle of the 50% of data values

- $IQR = Q_3 - Q_1$
- Q_1 is the midpoint of the lower half of the data values
- Q_2 is the median
- Q_3 is the midpoint of the upper half of the data values



Choosing the best measure of the centre of distribution

- Symmetric Distribution w/ no outliers >>> Range or IQR
- Skewed and/or outliers >>> IQR

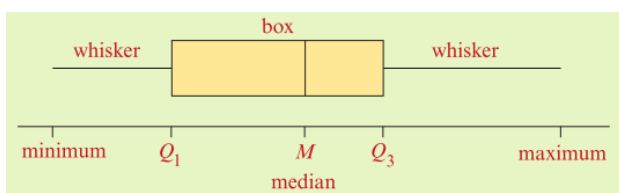
Five Number Summary:

minimum, Q_1 , median, Q_3 , maximum

* Includes outliers *

Box Plots

- A graphical display of a five-number summary
- **Note:** label the number line and box plot

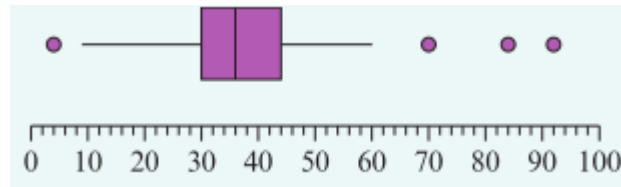


Outliers: $Q_3 + (1.5 \times IQR)$ or $Q_1 - (1.5 \times IQR)$

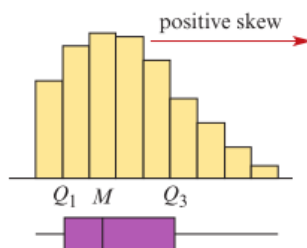
Example:

Construct a box plot given the five-number summary and outliers.

Minimum	4
First Quartile	30
Median	36
Third Quartile	44
Maximum	92
Outliers	4, 70, 84, 92



Positively skewed: $mean > median > mode$



Comparing Distributions

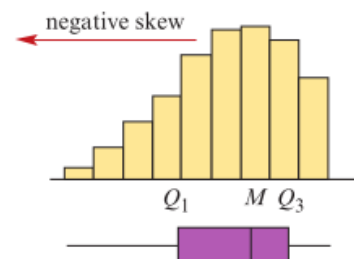
Shape: symmetrical or skewed, outliers

Centre: median, mean

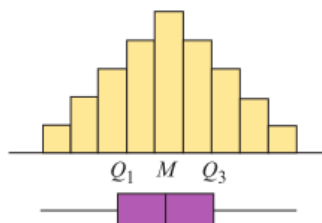
Spread: IQR, range, outliers

***ALWAYS QUOTE DATA**

Negatively skewed: $mean < median < mode$



Symmetric: $mean = median$



Measuring Centre of Distribution

Mean: the 'average' of a data set

- $Mean = \frac{\text{sum of data values}}{\text{number of data values}}$ or $\bar{x} = \frac{\Sigma x}{n}$
- Σ , 'sum of'
- x , to represent a data value
- \bar{x} , to represent the mean of the data values
- n , to represent the total number of data values.
- Measure of centre

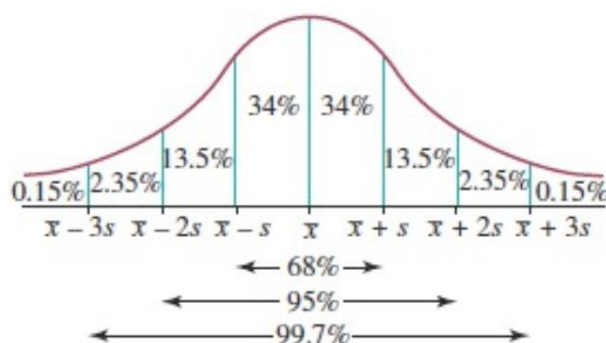
Mean vs Median

- Mean is the **BALANCE POINT** of the distribution
- Median is the **MIDPOINT** of the distribution

Choosing the best measure of the centre of distribution:

- **Symmetric Distribution w/ no outliers >>> Mean or Median** (approximately equal in value)
- **Skewed and/or outliers >>> Median** (mean is drastically changed due to outliers)
- The value of the median is relatively unaffected by the presence of extreme values in a distribution. For this reason, the median is frequently used as a measure of centre when the distribution is known to be clearly skewed and/or likely to contain outliers.

Normal Distribution and the 68-96-99.7% Rule (Standard Deviation)



68-96-99.7% Rule

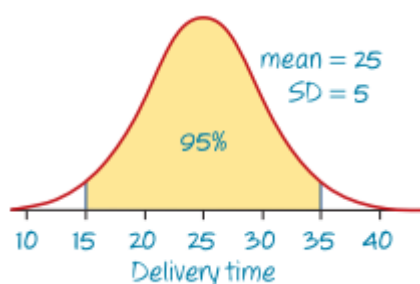
- 68% of the observations lie within one standard deviation of the mean
- 95% of the observations lie within two standard deviations of the mean
- 99.7% of the observations lie within three standard deviations of the mean.

Standard Deviation:

- $s = \sqrt{\frac{\sum(x-\bar{x})^2}{n-1}}$
- Measure of spread of the data
- Continuous data is almost symmetrical, or bell shaped

Example:

The distribution of delivery times for pizzas made by House of Pizza is approximately normal, with a mean of 25 minutes and a standard deviation of 5 minutes.



Standard (z) scores:

$$\text{standard score} = \frac{\text{actual score} - \text{mean}}{\text{standard deviation}} \quad \text{or} \quad z = \frac{x - \bar{x}}{s}$$

- Positive = above mean
- Negative = below mean
- Zero = equal to mean

Example:

The heights of a group of young women have a mean of $\bar{x} = 160$ cm and a standard deviation of $s = 8$ cm. Determine the standard or z-scores of a woman who is 150cm tall.

$$x = 150, \bar{x} = 160, s = 8$$

$$z = \frac{x - \bar{x}}{s} = \frac{150 - 160}{8} = -\frac{10}{8} = -1.25$$

Example:

The means and standard deviations for VCE Further Maths score of two schools are given below. Find the mark that has a z-score equally above Townson High mean as it is below City Secondary mean.

	Townson High School	City Secondary College
Mean	48.08	89.84
Standard Deviation	16.64	11.33

$$z_1 = -z_2$$

$$\frac{x - 48.08}{16.64} = \frac{-x + 89.84}{11.33}$$

$$11.33x - 544.7464 = -16.64x + 1494.9376$$

$$27.97x = 2039.684$$

$$x = 72.92$$

\therefore The mark is 72.92

Population and Samples

- Population = Whole group
- Sample = Subset of the group
- Simple Random Sample = Every member of the group has an equal chance of being selected

	Mean	Standard Deviation
Population	M	σ_x
Sample	\bar{x}	s_x

Bivariate Data

Response Variable: the variable, which is being influenced, the dependent variable (y axis)

Explanatory Variable: the variable, which is influencing, the independent variable (x axis)

Note: When investigating the correlation between two variables, the Explanatory Variable is the variable we expect to explain or predict the value of the Response Variable

Example:

Of the following pairs of variables, which are response, and which are explanatory?

	Explanatory	Response
Amount of alcohol consumed and reaction time	Amount of Alcohol	Reaction Time
Distance travelled, and time taken	Distance Travelled	Time Taken
Heart disease and amount of fat in diet	Amount of Fat	Heart Disease
Hours worked per week and salary	Hours worked	Salary

Two Way Frequency Table

- A statistical tool used to investigate associations between two categorical variables

Example:

According to the results summarized in the table, is there an association between support for banning mobile phones in cinemas and the sex of the respondent?

<i>Ban mobile phones</i>	<i>Sex</i>	
	<i>Male</i>	<i>Female</i>
Yes	87.9%	65.8%
No	12.1%	34.2%
<i>Total</i>	100.0%	100.0%

Yes, the percentage of males in support of banning mobile phones in cinemas (87.9%) was much higher than for females (65.8%).

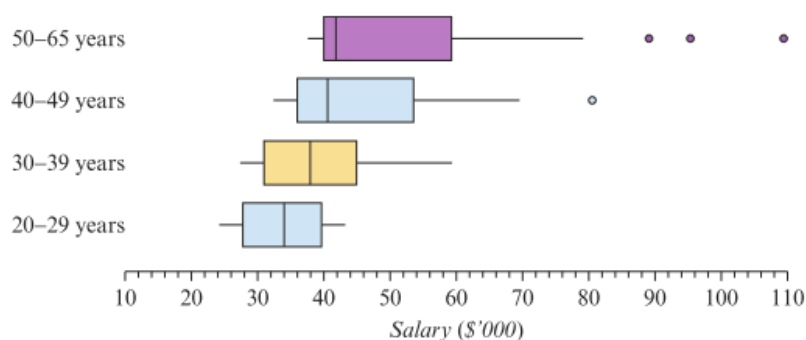
Note. A difference of 5% is significant

Parallel Box Plots

- A statistical tool used for investigating associations between a numerical and categorical variable

Example:

The parallel box plots below compare the salary distribution for four different age groups: 20–29 years, 30–39 years, 40–49 years and 50–65 years.



Identify and Describing Associations

- Median

Example:

The parallel box plots show that median salaries and age group are associated because median salaries increase with age group. For example, the median salary increased from \$34 000 for 20–29 year-olds to \$42 000 for 50–65 year-olds.

- IQR and/or ranges

Example:

From the parallel box plots we can see that the spread of salaries is associated with age group. For example, the IQR increased from around \$12 000 for 20–29-year-olds to around \$20 000 for 50–65-year-olds.

- Shape

Example:

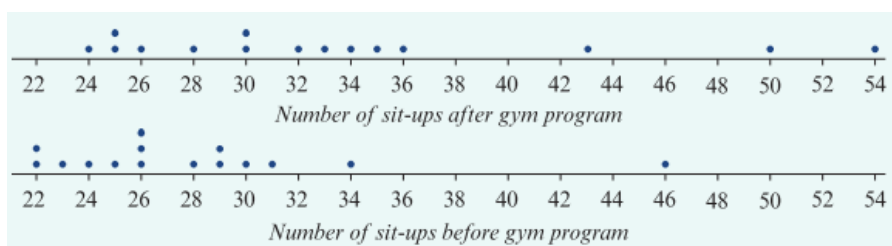
From the parallel box plots we can see that the shape of the distribution of salaries is associated with age group because of the distribution, which is symmetric for 20–29-year-olds, and becomes progressively more positively skewed as age increases. Outliers also begin to appear.

Parallel Dot Plots

- Used to investigate associations between numerical and categorical variables for small data sets

Example:

Do the parallel dot plots support the contention that the number of sit-ups performed is associated with completing the gym program? Write a brief explanation that compares medians.



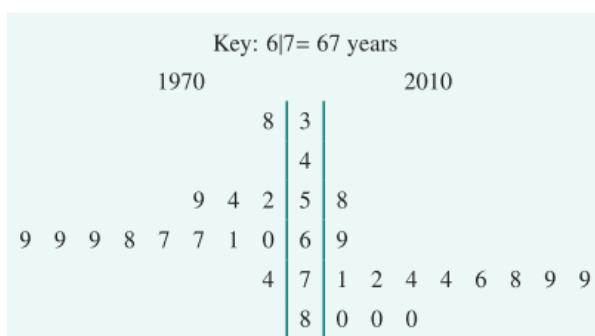
Yes; the median number of sit-ups performed after attending the gym program ($M = 32$) is considerably higher than the number of sit-ups performed before attending the gym program ($M = 26$). This indicates that the number of sit-ups performed is associated with completing the gym program.

Back to Back Stem Plots

- Used to investigate associations between numerical and categorical variables for small data sets

Example:

The back-to-back stem plot below displays the distribution of life expectancy (in years) for 13 countries in 2010 and 1970. Do the back-to-back stem plots support the contention that life expectancy is increasing over time? Write a brief explanation based on your comparisons of the two medians.



Yes: the median life expectancy in 2010 ($M = 76$ years) is considerably higher than the median life expectancy in 1970 ($M = 67$ years). This indicates that life expectancy is increasing over time.

Scatterplots

- Used to investigate associations between two numerical variables

Direction and Outliers >>> Positive, Negative, No association

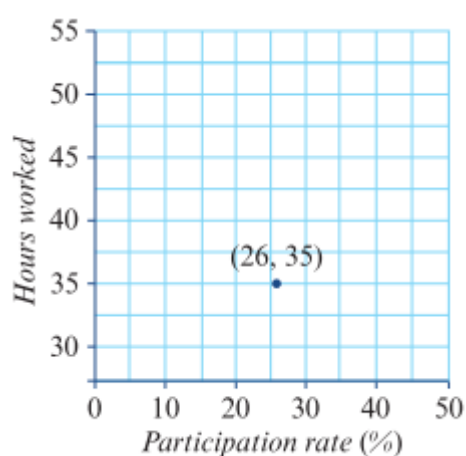
Form >>> Linear or Non-linear

Strength >>> Strong, Moderate, Weak, None

Example:

Construct a scatterplot using the data shown below.

Participation rate (%)	26	20	36	1	25	9	30	3	55
Hours worked	35	43	38	50	40	50	40	53	35



Which graph – two variables?

Response variable	Explanatory variable	Graph
Categorical	Categorical	Segmented bar chart Parallel bar chart Two-way frequency
Numerical	Categorical	Parallel box plot Parallel dot plot
Numerical	Categorical (two categories only)	Back-to-back stem plot Parallel box plot Parallel dot plot
Numerical	Numerical	Scatterplot

Pearson's Correlation Coefficient

- $r = \frac{\sum(x-\bar{x})(y-\bar{y})}{(n-1)S_x S_y}$
- Assumes that:
 - Variables are numeric
 - Association is linear
 - No outliers in the data set
- When converting r^2 to r , check whether the gradient is positive or negative

Strength of a Linear Relationship	
Strong positive association:	r between 0.75 and 0.99
Moderate positive association:	r between 0.5 and 0.74
Weak positive association:	r between 0.25 and 0.49
No association:	r between -0.24 and $+0.24$
Weak negative association:	r between -0.25 and -0.49
Moderate negative association:	r between -0.5 and -0.74
Strong negative association:	r between -0.75 and -0.99

Correlation of Determination

- Represented as r^2 , may be expressed as a decimal or percentage
- The coefficient of determination (as a percentage) tells us the variation in the response variable that is explained by the variation in the explanatory variable

Correlation and Causality:

- Correlation tells you about the strength of association instead of the source or cause
- Finding out if one variable causes the other variable to occur
- Causation cannot exist without correlation; correlation can exist without causation

Non-Casual Explanation for Association

Common Response: association with a common third variable

Confounding Variables: two possible explanations for association but no way to detangle their affects

Coincidence: association occurs by chance

Least Squares Regression Line

Fitting a straight line to bivariate data, minimising the sum of the squares of the residual

Residual: vertical distance between the actual data point and the regression line

- (Residual = Actual Data Value – Predicted Data Value)
- Takes into account every point on the scatterplot and is affected by outliers

The equation of the least squares regression line

The equation of the least squares regression line is given by $y = a + bx$,* where:

the **slope** (b) is given by: $b = \frac{r s_y}{s_x}$

and

the **intercept** (a) is then given by: $a = \bar{y} - b\bar{x}$

Here:

- r is the correlation coefficient
- s_x and s_y are the standard deviations of x and y
- \bar{x} and \bar{y} are the mean values of x and y

- When fitting a least squares regression line, it is assumed that:
 - Variables are numeric
 - Association is linear
 - No outliers in the data set

Interpreting the slope and the intercept of the regression line:

There is a **STRENGTH DIRECTION FORM** associations between RV and EV, ($r=?$).

Slope: On average, for every (one unit) increase in (x), (y) will (increase/ decrease) by (Gradient)

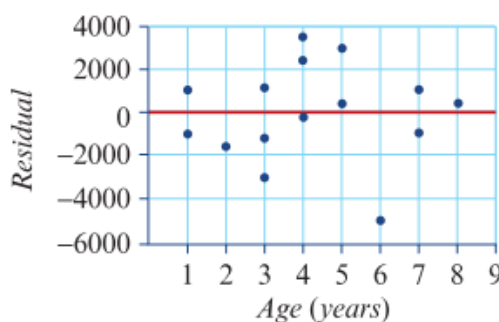
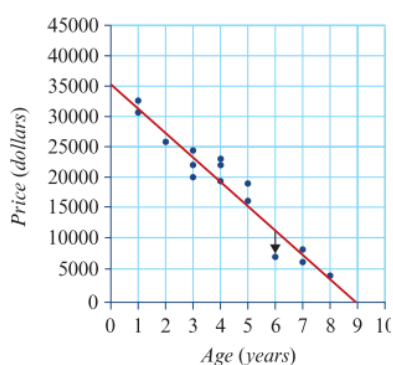
- **If slope is positive:** y increases as x increases
- **If slope is negative:** y decreases as x increases

Intercept: On average, (RV) is (Intercept) when (x) is 0

When using regression line to make predictions, substitute values into the equation

- **Interpolation:** predicting within the range of data, reasonably reliable
- **Extrapolation:** predictions outside the range of data, reasonable unreliable

Example:



Residual Plots

Linear: A random collection of points clustered around zero

Not Linear: A clear pattern

From the scatterplot we see that there is a strong negative, linear association between the price of a second hand car and its age, $r = -0.964$. There are no obvious outliers.

The equation of the least squares regression line is: $\text{price} = 35\,100 - 3940 \times \text{age}$.

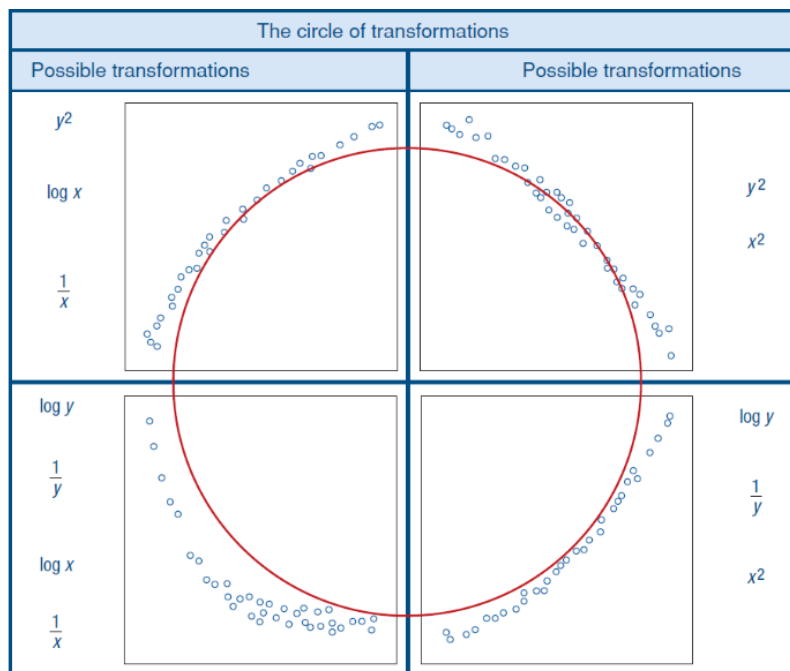
The slope of the regression line predicts that, on average, the price of these second-hand cars decreased by \$3940 each year.

The intercept predicts that, on average, the price of these cars when new was \$35 100.

The coefficient of determination indicates that 93% of the variation in the price of these second-hand cars is explained by the variation in their age.

The lack of a clear pattern in the residual plot confirms the assumption of a linear association between the price and the age of these second-hand cars.

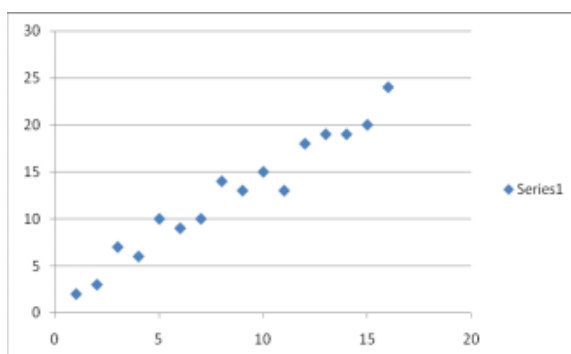
Transformations effects



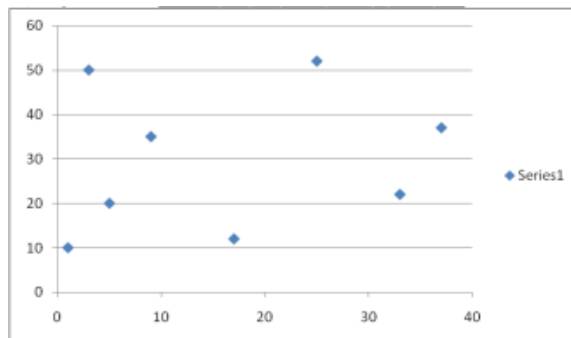
- Squared: stretches out the upper end of the scale on an axis
- Log: compresses the upper end of the scale on an axis
- Reciprocal: compress the upper end of the scale on an axis but to a greater extent than the log transformation
- **Note:** When transformations are applied include the transformed figure in the equation

Time Series Data:

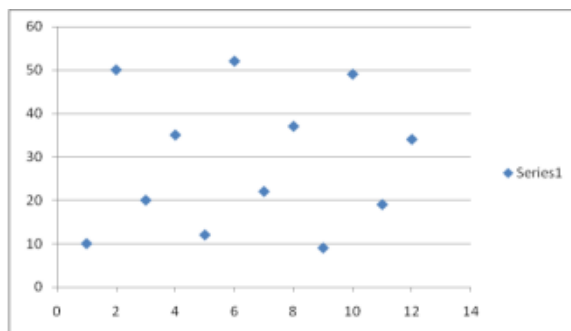
- Trend (Increasing or Decreasing): tendencies for values for values in a time series to generally increase or decrease over a significant period of time



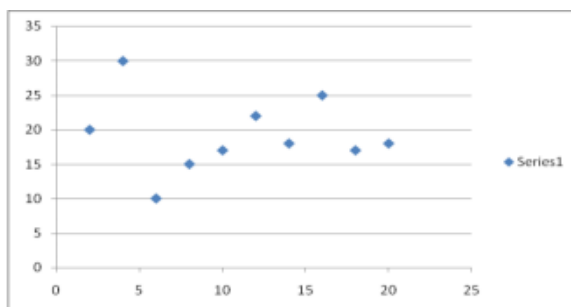
- Cycles (Clear Pattern): periodic movements in a time series, but over a period greater than a year



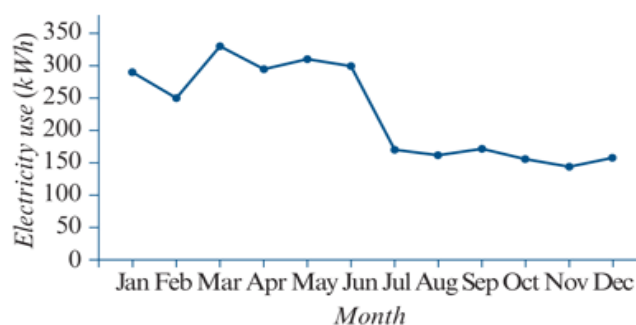
- Seasonal (Clear Pattern with equal spacing): periodic movement in a time series that has a calendar-related period – for example a year, a month or a week.



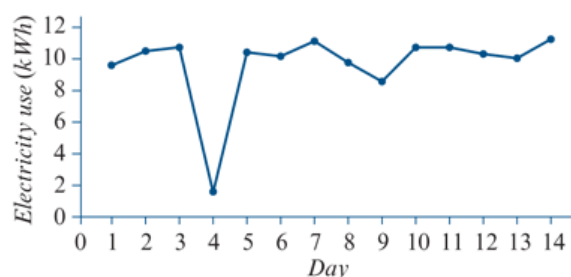
- Irregular (Random) Fluctuations: variations in a time series that we cannot reasonably attribute to systematic changes like trend, cycles, seasonality and structural change or an outlier.



- Structural Changes: sudden change in the established pattern of a time series plot



- Outliers: individual values that stand out from the general body of data



Smoothing: replacing individual data points in a time series to reduce random variation in data

Moving Mean Smoothing:

The three-moving mean

To use *three-moving mean smoothing*, replace each data value with the mean of that value and the values of its two neighbours, one on each side. That is, if y_1 , y_2 and y_3 are sequential data values, then:

$$\text{smoothed } y_2 = \frac{y_1 + y_2 + y_3}{3}$$

The first and last points do not have values on each side, so leave them out.

The five-moving mean

To use *five-moving mean smoothing*, replace each data value with the mean of that value and the two values on each side. That is, if y_1 , y_2 , y_3 , y_4 , y_5 are sequential data values, then:

$$\text{smoothed } y_3 = \frac{y_1 + y_2 + y_3 + y_4 + y_5}{5}$$

The first two and last two points do not have two values on each side, so leave them out.

Note: To decide best number of groups for smoothing, count data values until trend changes

For two and four moving mean with centring:

Centring: $\frac{\text{Mean 1} + \text{Mean 2}}{2}$

t	1	2	3	4	5	6	7	8	9	10
y	12	15	10	11	9	14	18	20	16	12
2-pt										
centre										

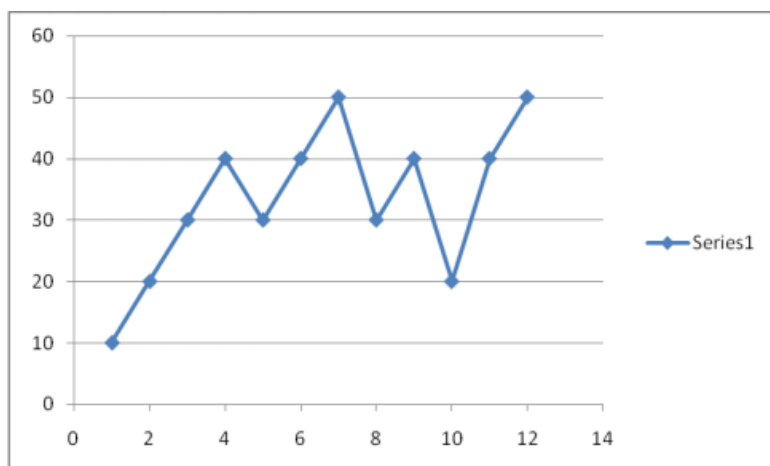
Smoothing Time Series using Moving Medians

A way of smoothing time series quickly and graphically

Step 1. Identify middle x values of points

Step 2. Identify middle y-value of points

Step 3. Mark point where these two values reside



Seasonal indices:

$$\text{seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

- Sum = number of seasons
- Average is one
- Interpreting: convert to percentage (eg. 1.2 = 120%: is 20% higher than monthly/season average)
- Example:

A season index for unemployment for February is 1.2.

A season index of 1.2 tells during February tells us that Unemployment figures for February are 20% above the monthly average.

Deseasonalisation: account for effects of seasonality in a time series

$$\text{deseasonalised figure} = \frac{\text{actual}}{\text{seasonal index}}$$

Correcting seasonality:

$$\frac{1}{\text{seasonal index}}$$

Example:

Mikki runs a shop and she wishes to determine quarterly seasonal indices based on last year's sales, which are shown in the table opposite.

Summer	Autumn	Winter	Spring
920	1085	1241	446

$$\text{Seasonal index} = \frac{\text{value for season}}{\text{seasonal average}}$$

$$\begin{aligned}\text{Quarterly average} &= \frac{920 + 1085 + 1241 + 446}{4} \\ &= 923\end{aligned}$$

$$SI_{\text{Summer}} = \frac{920}{923} = 0.997$$

$$SI_{\text{Autumn}} = \frac{1085}{923} = 1.176$$

$$SI_{\text{Winter}} = \frac{1241}{923} = 1.345$$

$$SI_{\text{Spring}} = \frac{446}{923} = 0.483$$

$$\text{Check: } 0.997 + 1.176 + 1.345 + 0.483 = 4.001$$

Seasonal indices

Summer	Autumn	Winter	Spring
0.997	1.176	1.345	0.483

Fitting a trend line:

- Least squares regression
- *Convert years to numbers* and have an **effective zero** (ie. 2012 = 0)
- Seasonal usually is not linear

Forecasting: predict values in the future (based off trend line)

Forecasted value needs to be reseasonalised:

$$\text{Seasonal forecast} = \text{deasonalised forecast} \times \text{seasonal index}$$

The seasonal index for heaters in winter is 1.25.

To correct for seasonality, the actual heater sales in winter should be:

- A: reduced by 20%
- B: increased by 20%
- C: reduced by 25%
- D: increased by 25%
- E: increased by 75%

To correct for seasonality: achieve a seasonal index of one

If the seasonal index is over one, the deseasonalised value is less; thus actual sales is reduced.

To find percentage;

$$\frac{1.25 - 1}{1.25} \times 100 = 20$$

OR

$$\frac{1}{1.25} = 0.8 \quad \therefore 1 - 0.8 = 0.2$$

Since 0.8 is less than one, it is a decrease. Therefore, decrease of 20%.

Describe the general pattern in rainfall that is revealed by the smoothed time series plot.

Until April, there is an increase in monthly rainfall. It then remains relatively constant for the remainder of the year.

Recursion and Financial Modelling

Sequences: list of numbers written down in succession, with each number being a 'term'

Recurrence Relations: mathematical rule used to generate a sequence

$$V_0 = \text{Initial Term}, V_n + 1 = v_n \pm D$$

starting value rule

Example:

Generate the first five term of the sequence, showing the values of the first four iterations (each time a rule is applied).

$$V_0 = 9, V_n + 1 = V_n - 4$$

$$V_0 = 9, V_1 = 5, V_2 = 1, V_3 = -3, V_4 = -7$$

\therefore Sequence is 9, 5, 1, -3, -7

Modelling Linear Growth and Decay: when a quantity increases or decreases by the same amount at regular intervals

$$V_n + 1 = V_n + D \gg \text{Linear Growth}$$

$$V_n + 1 = V_n - D \gg \text{Linear Decay}$$

Simple Interest

Interest: fee received or paid for lending or borrowing money

Principal: initial amount borrowed or invested

- **Simple Interest Loans:** borrow money from a bank and pay a fixed amount of interest to the bank
- **Simple Interest Investments:** deposit money in a bank account and received a fixed amount of interest from the bank
- **Note:** must have fixed amount and occur over regular time periods

Let V_0 be the initial value of the simple interest investment or loan.

Let r be the annual interest rate.

The value of a simple interest investment or loan after n years is

$$V_n = V_0 + nD \quad \text{where} \quad D = \frac{r}{100} V_0 \quad \text{or} \quad V_n = V_0 + n \times \frac{r}{100} \times V_0$$

Depreciation: reductions of an item's value over time

Future Value: estimated value of an item after a certain period of time through taking into account its potential depreciation

Scrap Value: when an item is no longer an asset and depreciated to a point where it is disposed or sold

- **Flat Rate Depreciation:** constant amount is subtracted to decay the value of the asset after every time period

Let V_0 be the initial value of the asset.

Let r be the flat rate of depreciation.

The value of the asset after n years is

$$V_n = V_0 - nD \quad \text{where} \quad D = \frac{r}{100} V_0 \quad \text{or} \quad V_n = V_0 - n \times \frac{r}{100} \times V_0$$

- **Unit Cost Depreciation:** items losing value because of how often they are used, rather than its age

Let V_0 be the initial value of the asset.

Let D be the cost per unit of use.

The value of the asset after n units of use is

$$V_n = V_0 - nD$$

Modelling Geometric Growth and Decay: when quantities increase or decrease by the same percentage at regular intervals

$V_{n+1} = RV_n$ for $R > 1 \gg$ Geometric Growth

$V_{n+1} = RV_n$ for $0 < R < 1 \gg$ Geometric Decay

Compound Interest: any interest earned after one time period is added to the principal and then contributes to the earned interest in the next time period

Note: to determine interest rate which is not compounded annually through division. For example, if it is compounded monthly, divide by 12

- **Compound Interest Investment and Loans**

Let V_n be the value of the investment after n years.

Let r be the percentage interest per compound period.

The recurrence model for the value of the investment after n compounding periods is:

$$V_0 = \text{principal}, V_{n+1} = RV_n \quad \text{where} \quad R = 1 + \frac{r}{100}$$

- **Reducing Balance Depreciation:** form of depreciation, where the value of an asset decays geometrically. Each year, the value will be reduced by a percentage, $r\%$, of the previous year's value.

Let V_n be the value of the asset after n years.

Let r be the annual percentage depreciation.

The recurrence model for the value of the investment after n years is:

$$V_0 = \text{initial value, } V_{n+1} = RV_n \text{ where } R = 1 - \frac{r}{100}$$

Rules for nth term in a sequence

Used to predict the value of the n th term (iteration) in the sequence without generating all of the other terms.

A rule for individual terms of a geometric growth and decay sequence

For a geometric growth or decay recurrence relation

$$V_0 = \text{starting value, } V_{n+1} = RV_n$$

the value of the n th term of the sequences is generated by the rule:

$$V_n = R^n \times V_0$$

Compound interest loans and investments

Let V_0 be the amount borrowed or invested (principal).

Let r be the interest rate per compounding period.

The value of a compound interest loan or investment after n compounding periods, V_n , is given by the rule

$$V_n = \left(1 + \frac{r}{100}\right)^n \times V_0$$

Reducing-balance depreciation

Let V_0 be the purchase price of the asset.

Let r be the annual percentage rate of depreciation.

The value of an asset after n years, V_n is given by the rule

$$V_n = \left(1 - \frac{r}{100}\right)^n \times V_0$$

Nominal Interest Rates: compound interest rates quoted as annual rates or interest rates per annum

Usually the nominal rate is stated, then a different time period is mentioned for which the interest is calculated and paid. This is called the 'compounding period'

It is assumed that:

- 12 equal months in every year
- 4 quarters in every year
- 26 fortnights in a year
- 52 weeks in a year
- 365 days in a year

Note: To convert nominal interest rate to compounding interest rate, divide by its respective numbers, eg. Covert 3.6% per annum to a monthly rate. $\frac{3.6}{12} = 0.3\%$

Effective Interest Rates: interest earned after one year expressed as a percentage of the amount borrowed or invested

- Used to compare investment performance

The effective interest rate of a loan or investment is the interest earned after one year expressed as a percentage of the amount borrowed or invested.

Let:

- r be the nominal interest rate per annum
- $r_{\text{effective}}$ be the effective annual interest rate
- n be the number of times the interest compounds each year.

The effective annual interest rate is given by: $r_{\text{effective}} = \left(\left(1 + \frac{r/n}{100} \right)^n - 1 \right) \times 100\%$

Example:

Brooke would like to borrow \$20000. She is deciding between two loan options:

- option A: 5.95% per annum compounding weekly
- option B: 6% per annum compounding quarterly.

Option A

$$r = 5.95$$

There are 52 weeks in a year so
 $n = 52$.

$$\begin{aligned} r_{\text{effective}} &= \left(\left(1 + \frac{5.95/52}{100} \right)^{52} - 1 \right) \\ &\quad \times 100\% \\ &= 6.13\% \end{aligned}$$

Option B

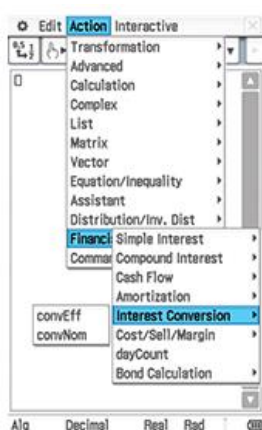
$$r = 6$$

There are 4 quarters in a year
so $n = 4$.

$$\begin{aligned} r_{\text{effective}} &= \left(\left(1 + \frac{6/4}{100} \right)^4 - 1 \right) \\ &\quad \times 100\% \\ &= 6.14\% \end{aligned}$$

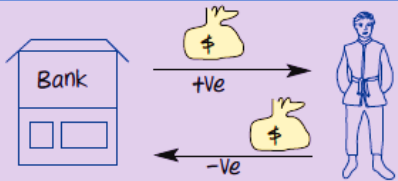
Brooke is borrowing money, so the best option is the one with the lowest effective interest rate. She will pay less interest with option A.

USING CAS



convEff (Compounds per
year, Nominal IR %)

Modelling and Analysing Reducing Balance Loans and Annuities

 <div> Rules Bank gives you money: positive (+ve) You give bank money: negative (-ve) </div>		
Loan	PV positive PMT negative FV 0 or negative	Bank gives (lends) you money. You pay off the loan by making regular payments to the bank. At the end of the loan, it is either fully paid out ($FV = 0$) or you still have to pay money to the bank.
Annuity	PV negative PMT positive FV 0 or positive	You give money to the bank. The bank makes regular payments to you from the investment. At the end of the annuity, there is either no money left ($FV = 0$) or the bank pays you back what is left.
Investment that is added to on a regular basis	PV negative PMT negative FV positive	You give money to the bank. You increase the amount you have invested by making regular payments to the bank. At the end of the investment the bank pays you the total final value of your investment.

Financial Solver

N	total no. of payments
I%	<u>annual</u> interest rate
PV	present value of the loan
PMT	amount paid at each payment
FV	future value
P/Y	no. of payments <u>per year</u>
C/Y	no. times interest is compounded <u>per year</u>

Combining Geometric Growth and Decay

$$V_0 = \text{Starting Value}, V_{n+1} = RV_n \pm D$$

Example:

The number of trout in a fish farm pond after n months, T_n , can be modelled using the recurrence relation: $T_0 = 10000, T_{n+1} = 1.1 \times T_n - 3000$. After how many months will there be no trout left in the pond?

10000	10000
$10000 \cdot 1.1 - 3000$	8000
$8000 \cdot 1.1 - 3000$	5800
$5800 \cdot 1.1 - 3000$	3380
$3380 \cdot 1.1 - 3000$	718
$718 \cdot 1.1 - 3000$	-2210.2

\therefore There will be no trout left in the pond after 5 months

Reducing Balance Loans: compound interest loan with regular payments

Let V_n be the *balance* of the loan after n payments have been made.

Let r be the *interest rate* per compounding period.

Let D be the *payment* made.

A recurrence relation rule that can be used to model a reducing balance loan is

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n - D$$

where $R = 1 + \frac{r}{100}$

Example:

Alyssa's loan can be modelled by the recurrence relation:

$V_0 = 1000, V_{n+1} = 1.0125V_n - 257.85$, What is the balance of the loan (the amount she still owes) after she has made two payments? Give your answer to the nearest cent.

1000	1000
$1000 \cdot 1.0125 - 257.85$	754.65
$754.65 \cdot 1.0125 - 257.85$	506.233

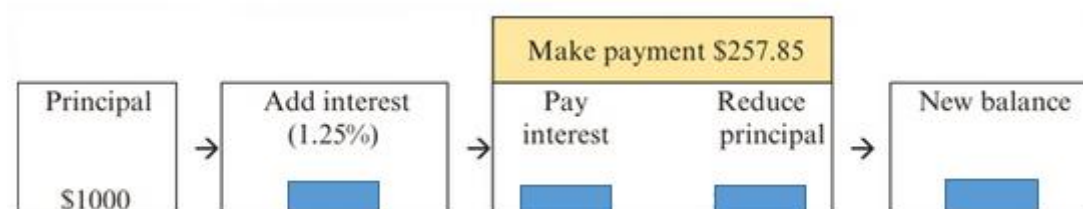
\therefore The balance of the loan is \$506.23

Amortisation Loans: loans that are repaid by making regular payments until the balance is zero

- Part of each payment goes towards the *interest* owed on the unpaid balance.
- Remainder of payment used to reduce the unpaid balance.

Example:

Amortisation Process



Example:

Amortisation Table

Payment number	Payment amount	Interest paid	Principal reduction	Balance of loan
0	0	0	0	1000.00
1	257.85	12.50	245.35	754.65
2	257.85	9.43	248.42	506.23
3	257.85	6.33	251.52	254.71
4	257.89*	3.18	254.71	0.00
Total		31.44	10000.00	

Properties of a reducing balance loan

1. interest paid = interest rate per payment period \times unpaid balance.
2. principal reduction = payment made - interest paid
3. balance of loan = balance owing - reduction in balance
4. cost of repaying the loan = the sum of the payments
5. total interest paid = total cost of repaying the loan – principal
6. The amount of interest paid from each payment decreases with payment number while the amount of principal repaid increases.

Interest Only Loans: borrower only repays the interest which is charged, the value of the loan remains the same

Let V_0 be the *principal* of the loan.

Let r be the interest *rate* per compounding period.

Let D be the regular payment per compounding period.

For interest-only loans:

payment = interest charged

so
$$D = \frac{r}{100} \times V_0$$

$D = \frac{r}{100} V_0$ Value of the Loan is the Same

$D > \frac{r}{100} V_0$ Value of the Loan Decreases

$D < \frac{r}{100} V_0$ Value of the Loan Increases

Example:

Jane borrows \$50000 to buy some shares. Jane negotiates an interest-only loan for this amount, at an interest rate of 9% per annum, compounding monthly. What is the monthly amount Jane will be required to pay?

$$V_0 = 50\,000$$

$$r = 0.75\%$$

$$D = \frac{0.75}{100} \times V_0$$

$$D = \frac{0.75}{100} \times 50\,000$$

$$D = 375$$

Jane will need to repay \$375 every month on this interest-only loan.

Annuities: form of investment, entitling the investors to a series of regular payments

Let r be the interest *rate* per compounding period.

Let D be the payment received.

A recurrence relation that can be used to model the value of an annuity after n payments, V_n , is

$$V_0 = \text{principal}, V_{n+1} = RV_n - D$$

where $R = 1 + \frac{r}{100}$

Example:

Reza plans to travel overseas for 6 months. He invests \$12000 in annuity that earns interest at the rate of 6% per annum, providing him with a monthly income of \$2035 per month for 6 months.

Model this annuity using a recurrence relation of the form:

$$V_0 = \text{the principal}, V_{n+1} = RV_n - D$$

where V_n is the value of the annuity after n payments have been received.

Let V_n be the value of the annuity after n payments have

$$V_0 = 12\,000 \quad i. \quad r = \frac{6\%}{12} = 0.5\% \text{ or } 0.005 \quad R = 1.005 \quad D = 2035$$

$$V_0 = 12\,000 \quad V_{n+1} = 1.005V_n - 2035$$

Example:

Amortisation Table for Annuity

Payment number	Payment received	Interest earned	Principal reduction	Balance of annuity
0	0	0.00	0.00	12000.00
1	2035.00	60.00	1975.00	10025.00
2	2035.00	50.13	1984.88	8040.13
3	2035.00	40.20	1994.80	6045.33
4	2035.00	30.23	2004.77	4040.55
5	2035.00	20.20	2014.80	2025.76
6	2035.88	10.13	2024.87	0.00

Properties of annuities

1. interest earned = interest rate per compounding period \times balance.
2. principal reduction = payment made – interest earned
3. balance of loan = previous balance – principal reduction
4. total return from the annuity = the sum of the payments made.
5. total interest earned = total payments received – principal

Perpetuities: type of annuity where the payment is the same as interest

Let V_0 be the initial value of the annuity.

Let r be the interest *rate* per compounding period.

Let D be the regular payment received.

For perpetuities:

$$\text{payment received} = \text{interest earned}$$

so

$$D = \frac{r}{100} \times V_0$$

Example:

Elizabeth invests her superannuation payout of \$500000 into a perpetuity that will provide a monthly income without using any of the initial investment.

If the interest rate for the perpetuity is 6% per annum, what monthly payment will Elizabeth receive?

Elizabeth invests \$500 000 so $V = 500\,000$.

The annual interest rate is 6% so $r = \frac{6}{12} = 0.5\%$.

$$D = \frac{r}{100} \times V_0$$

$$D = \frac{0.5}{100} \times 500\,000$$

$$D = 2500$$

Elizabeth will receive \$2500 every month from her investment.

Annuity investments: compound interest investment with regular additions to the principal

- Increase the rate at which investment grows by making additional payments on a regular basis

Let r be the *interest rate* per compounding period.

Let D be the *payment* made.

A recurrence relation that can be used to model the value of a compound interest investment, V_n , after n additional payments have been made is

$$V_0 = \text{the principal}, \quad V_{n+1} = RV_n + D$$

where $R = 1 + \frac{r}{100}$.

Example:

Nor plans to travel overseas when she finishes her VCE. She has already saved \$1200 and thinks that she can save an additional \$50 each month that she plans to add to her savings account. The account pays interest at a rate of 3% per annum, compounding monthly.

Model this investment using a recurrence relation of the form

$V_0 = \text{the principal}$, $V_{n+1} = RV_n + D$ where V_n is the value of the investment after n payments (additions to the principal) have been made.

Let V_n be the value of the annuity after n payments.

$$V_0 = 1200$$

$$r = \frac{3\%}{12} = 0.25\% \text{ or } 0.0025$$

$$V_0 = 1200$$

$$V_{n+1} = 1.0025V_n + 50$$

Example:

Amortisation Table for Annuity Investment

Payment number	Payment made	Interest earned	Principal increase	Balance of investment
0	0.00	0.00	0.00	1200.00
1	50.00	3.00	53.00	1253.00
2	50.00	3.13	53.13	1306.13
3	50.00	3.27	53.27	1359.40
4	50.00	3.40	53.40	1412.80
5	50.00	3.53	53.53	1466.33
6	50.00	3.67	53.67	1519.99
7	50.00	3.80	53.80	1573.79
8	50.00	3.93	53.93	1627.73
9	50.00	4.07	54.07	1681.80
10	50.00	4.20	54.20	1736.00
11	50.00	4.34	54.34	1790.34
12	50.00	4.48	54.48	1844.82

Properties of an Annuity Investment

1. interest earned = interest rate per compounding period \times previous balance
2. principal increase = payment made + interest earned
3. Balance of investment = previous balance + interest + payment made
4. total interest earned = balance of loan – (principal + additional payments)